8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 1

1: NON-ABELIAN GAUGE THEORIES

In 8.323, we have studied the quantum theory of:

1. Scalar fields, spin-0,
2. Dirac fields, spin-$\frac{1}{2}$,
3. Maxwell fields, spin-1.

The last two fields are the content of quantum electrodynamics, an extremely successful description of the electromagnetic interaction observed in nature. For the weak and strong nuclear interactions, however, the fields listed above are not sufficient. We require non-Abelian gauge theories. In this course, we will describe the formulation and quantization of these fields. In 8.325, we will learn how to use them to describe the weak and strong interactions.

1.1: SYMMETRIES, LIE GROUPS AND LIE ALGEBRAS

Noether's theorem reveals a correspondence between symmetries and conserved quantities:

$$\begin{aligned} \text{Symmetry} &\longleftrightarrow \text{Conserved quantity},\\ \text{Spatial translation} &\longleftrightarrow \text{Momentum},\\ \text{Temporal translation} &\longleftrightarrow \text{Energy},\\ \text{Rotation} &\longleftrightarrow \text{Angular momentum},\\ \text{Internal symmetries} &\longleftrightarrow \text{Charges}. \end{aligned}$$

Example 1: Dirac spinor field

$$\mathscr{L} = -i\overline{\psi}(\gamma^\mu \partial_\mu - m)\psi. \tag{1}$$

Note that spinor indices will always be suppressed, and that we use the $(-+++)$ metric convention. This Lagrangian is invariant under the internal symmetry transformation $\psi(t,\vec{x}) \longrightarrow e^{i\alpha}\psi(t,\vec{x})$. The conserved charge associated with this symmetry is $Q = \int d^3x\, \psi^\dagger \psi$.

Remarks:

1. All possible phase multiplications $e^{i\alpha}$ form a group, called $U(1)$. $U(1)$ is Abelian, that is, $e^{i\alpha_1}e^{i\alpha_2} = e^{i\alpha_2}e^{i\alpha_1}$. A **Group** is a set closed under a multiplication, satisfying the axioms of (i) associativity, (ii) existence of an identity and (iii) existence of an inverse.
2. $U(1)$ is a continuous group. $U(1) = S^1$, the unit circle, as a manifold. A **Lie Group** is a group that is also a differential manifold.

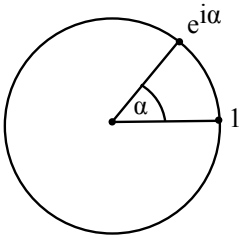

Figure 1: $U(1)$ as the manifold S^1 with the elements 1 and $e^{i\alpha}$ shown explicitly.

Example 2: Two Dirac fields

$$\mathscr{L} = -i\overline{\psi}_1(\gamma^\mu\partial_\mu - m_1)\psi_1 - i\overline{\psi}_2(\gamma^\mu\partial_\mu - m_2)\psi_2. \tag{2}$$

This Lagrangian is invariant under the internal symmetry transformations

$$U(1) : \psi_1(x) \longrightarrow e^{i\alpha_1}\psi_1(x),\ U(1) : \psi_2(x) \longrightarrow e^{i\alpha_2}\psi_2(x). \tag{3}$$

The group formed by this set of transformations is $U(1) \times U(1)$, with distinct charges of ψ_1 and ψ_2. If $m_1 = m_2 = m$, there is a larger symmetry, since we can mix ψ_1 and ψ_2. Introduce $\Psi = \binom{\psi_1}{\psi_2}$, $\bar{\Psi} = \left(\overline{\psi}_1\ \overline{\psi}_2\right)$. We can then rewrite (2) as

$$\mathscr{L} = -i\overline{\Psi}(\gamma^\mu\partial_\mu - m)\Psi, \tag{4}$$

which is invariant under the internal symmetry transformation

$$U(2) : \Psi(x) \longrightarrow \mathbf{U}\boldsymbol{\Psi}(x), \bar{\Psi}(x) \longrightarrow \bar{\boldsymbol{\Psi}}(x)\mathbf{U}^\dagger, \tag{5}$$

where $\mathbf{U}$ is a 2×2 matrix such that

$$\mathbf{U}\mathbf{U}^\dagger = \mathbf{1}. \tag{6}$$

The complex matrices satisfying this condition form the group $U(2)$. We can separate out an overall phase rotation in these transformations:

$$\Psi(x) \longrightarrow \mathbf{e}^{\mathbf{i}\alpha}\Psi(x) \tag{7}$$

where $\mathbf{U} = \mathbf{e}^{\mathbf{i}\alpha} = e^{i\alpha}\mathbb{I}_{2\times 2}$. We can constrain the residual transformation $\mathbf{U}'$ to satisfy

$$\det \mathbf{U}' = 1. \tag{8}$$

The set of 2×2 matrices satisfying (6) and (8) form the group $SU(2)$. In summary, for $m_1 = m_2$, (2) is invariant under

$$U(2) = SU(2) \times U(1) \tag{9}$$

where $U(1)$ is proportional to the identity matrix and commutes with $SU(2)$.

Remarks:

1. $SU(2)$ is a non-Abelian group. $\mathbf{U_1U_2} \neq \mathbf{U_2U_1}$. It has three continuous parameters (diffeomorphic as a manifold to S^3).

2. Any $SU(2)$ group element can be written as $\mathbf{U} = \exp\left[i\Lambda_a \frac{\sigma_a}{2}\right]$, with Λ_a constant, a =1, 2, 3. σ_a here are the Pauli matrices, and $\frac{\sigma_a}{2}$ are the generators of $SU(2)$, satisfying the associated $\mathfrak{su}(2)$ Lie algebra commutation relations,
$$\left[\frac{\sigma_a}{2}, \frac{\sigma_b}{2}\right] = i\epsilon_{abc}\frac{\sigma_c}{2}. \tag{10}$$

3. For infinitesimal Λ_a, $\mathbf{U}$ =1+$i\Lambda_a \frac{\sigma_a}{2} + \ldots$, we have that $\delta\Psi = i\Lambda_a \frac{\sigma_a}{2}\Psi$, and by Noether's prescription, we find an associated set of conserved charges Q_a, $(a = 1, 2, 3)$, which satisfy the commutation relations $[Q_a, Q_b] = i\epsilon_{abc}Q_c$ (see problem set). That is, the conserved charges satisfy the Lie algebra commutation relations.

4. Starting with n Dirac fields of the same mass, we find the symmetry group $U(n) = SU(n) \times U(1)$, where $U(n)$ is the set of $n \times n$ matrices under matrix multiplication that satisfy $\mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger\mathbf{U} = \mathbf{1}$. $SU(n)$ and $U(n)$ are examples of Lie groups.

1.1.2. More information on Lie groups and Lie algebras

(for a more detailed treatment, the reader is referred to the notes by B. Zweibach, *Peskin and Schroeder*, §15.4, and *Weinberg, Volume I*, p53-55, *Volume II*, §15.1-15.2.):

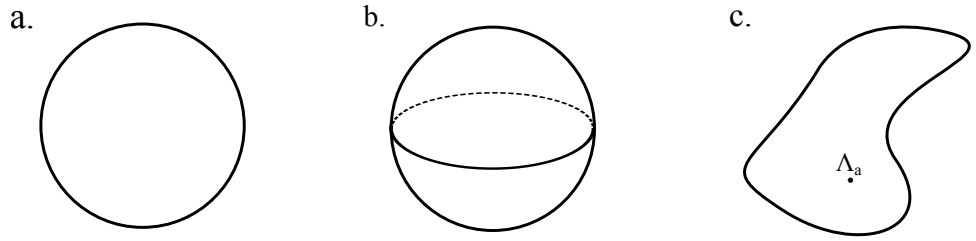

Figure 2: a. $U(1)$, the manifold S^1. b. $SU(2)$, the manifold S^3 (one dimension suppressed). c. An arbitrary manifold with a point Λ_a labelled.

Lie group: a continuous group that is also a differential manifold. Examples include $U(1)$, which is the manifold S^1, and $SU(2)$, which is the manifold S^3. The group elements are labelled $g(\Lambda_a) \in G$, where Λ_a are parameters labelling points of the manifold, $a = 1, 2, \ldots, \dim G$. They satisfy a group multiplication:
$$g(\Lambda) \circ g(\Lambda') = g(\Lambda''), \tag{11}$$
where
$$\Lambda''_a = f_a(\Lambda, \Lambda'). \tag{12}$$

Lie groups are tailor-made for describing continuous symmetry transformations in physics: the group product corresponds to the composition of transformations. The group product satisfies the properties of (i) closure, (ii) associativity, (iii) existence of an inverse element, and (iv) existence of an identity element. Now consider a system described by some Lagrangian, $\mathscr{L}$. All physical symmetric transformations form Lie groups. Examples include:

$\mathbb{R}^{1,3}$- space-time translations,

$SO(1,3)$ - Lorentz boosts and rotations,

$U(n)$ - internal symmetry groups.

A **compact** Lie group is a manifold of finite volume.

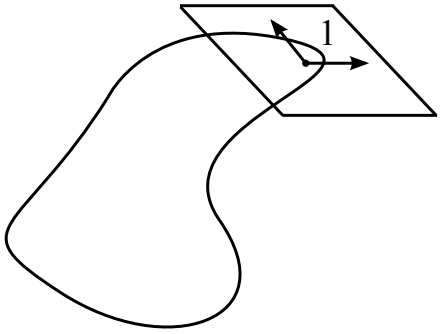

Figure 3: A Lie algebra as the elements in the infinitesimal vicinity of the group identity element.

Lie algebra: as physicists, we are often interested in *infinitesimal* transformations, in order to find the conserved currents. As mathematicians, describing the general structure of a Lie group is difficult. The trick is to consider elements of G that differ infinitesimally from the identity element, from which one may build up the entire group. Choose Λ_a so that $g(\Lambda_a = 0) = 1$. Then

$$g(\epsilon_a) = 1 + i\epsilon_a T^a + O(\epsilon^2), \tag{13}$$

where $a = 1, \ldots, \dim G$. It is a physicist's convention to extract a factor of i explicitly in the expansion, so that the generators are hermitian for unitary group transformations. ϵ_a parameterizes the tangent space around $g = 1$, with T^a a basis for the tangent space. Mathematically, $\epsilon_a T^a \in T_1 G$. Putting (12) into (13), we find T^a should satisfy

$$\left[T^a, T^b\right] = i f^{cab} T^c, \tag{14}$$

where $\left[T^a, T^b\right] \equiv T^a \circ T^b - T^b \circ T^a$, defining the Lie algebra $\mathfrak{g}$ associated with the Lie group G. The f^{cab} are the structure constants of the Lie algebra and are necessarily anti-symmetric in a, b. There is another constraint on the structure constants: as the group multiplication, $\circ$, is associative, $\left[T^a, T^b\right]$ satisfies the Jacobi identity:

$$\left[\left[T^a, T^b\right], T^c\right] + \left[\left[T^b, T^c\right], T^a\right] + \left[\left[T^c, T^a\right], T^b\right] = 0. \tag{15}$$

Hence,

$$f^{dab} f^{edc} + f^{dca} f^{edb} + f^{dbc} f^{eda} = 0. \tag{16}$$

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Lecture 2

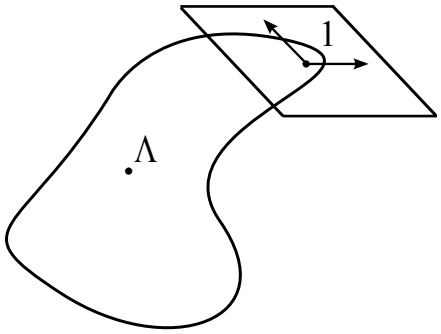

Figure 1: An element Λ of the manifold of the Lie group G, and the Lie algebra $\mathfrak{g}$ as the tangent space of the identity element.

Some facts about Lie groups and Lie algebras:

1. Different Lie groups can have the same Lie algebra. The Lie algebra determines the Lie group up to discrete choices of global structure. For example, $SU(2) = S^3$, $SO(3) = S^3/\mathbb{Z}_2$.

2. An **invariant subalgebra** is a subset of a Lie algebra $\mathfrak{g}' \subset \mathfrak{g}$ which is closed under the action of $\mathfrak{g}$. That is, $[\mathfrak{g}, \mathfrak{g}'] \subset \mathfrak{g}'$. A **simple** Lie algebra is a Lie algebra which does not contain invariant subalgebras and which is not Abelian. The complex simple Lie algebras are completely classified: $\mathfrak{su}(n)$, $\mathfrak{so}(2n)$, $\mathfrak{so}(2n+1)$, $\mathfrak{sp}(n)$, $E_{6,7,8}$, F_4 and G_2 are the only possibilities.

3. For a compact Lie group, it is always possible to choose a basis of T_a so that $f_{abc} = f^a_{bc}$ is truly antisymmetric (there is no distinction between upper and lower indices). All internal symmetry groups are compact. For example, $SU(n)$ (the set of $n \times n$ unitary matrices):

$$U = \exp\left[i\Lambda^a T_a\right], \ a = 1, \ldots, n^2 - 1, \tag{1}$$

where

$$\mathrm{Tr}(T_a) = 0, \ (T_a)^\dagger = T_a, \tag{2}$$

that is, the generators are hermitian and traceless, and hence we can choose

$$[T_a, T_b] = i f_{abc} T_c, \tag{3}$$

where the f_{abc} are fully antisymmetric.

4. Physically, for example considering $\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$ with an $SU(n)$ symmetry, we find a set of associated Noether charges $\hat{Q}_a$, $a = 1, \ldots n^2 - 1$, satisfying the Lie algebra commutation relations, $\left[\hat{Q}_a, \hat{Q}_b\right] = i f_{abc} \hat{Q}_c$. Then the transformations on Ψ are generated by the Noether charges:

$$\hat{U} = \exp\left[\Lambda^a \hat{Q}_a\right], \tag{4}$$

where $\left[\epsilon^a \hat{Q}_a, \Psi\right] = \epsilon^a T_a \Psi$. That is,

$$\hat{U} \Psi \hat{U}^\dagger = U \Psi, \tag{5}$$

where U is given in (1). This is checked explicitly for $SU(2)$ in the problem set.

1.2: THE GAUGE PRINCIPLE (QUANTUM ELECTRODYNAMICS REVISITED)

Referring back to the $U(1)$ invariant Lagrangian we studied in lecture 1:

$$\mathscr{L} = -i\overline{\psi}(\gamma^\mu \partial_\mu - m)\psi, \tag{6}$$

which is symmetric under $\psi(t,\vec{x}) \longrightarrow e^{i\alpha}\psi(t,\vec{x})$, we note that for the Lagrangian to be symmetric, it is necessary that α is not position-dependent. That is, all spacetime points transform in the same way. The transformation is no longer a symmetry for general $\alpha = \alpha(x)$, that is, if we allow different phase rotations at different spacetime points. The mass term is invariant under these more general transformations. The kinetic term, however, transforms as

$$\partial_\mu \Psi \to \partial_\mu(e^{i\alpha(x)}\Psi(x)) = e^{i\alpha(x)}\partial_\mu(\Psi(x)) + i\partial_\mu(\alpha(x))e^{i\alpha(x)}\Psi(x), \tag{7}$$

where we have kept the x-dependence explicit. The second term is the problem. We want to construct a theory (i.e. a Lagrangian) which is invariant for a general $\alpha(x)$, that is, a theory with a local $U(1)$ symmetry. The answer involves the introduction of a new vector field, and leads to quantum electrodynamics, as studied in 8.323. This example, in fact, embodies a deep principle: the principle of gauge invariance. As we will discuss,

$$\begin{aligned} \text{Local symmetries} &\Rightarrow \text{Interactions,} \\ \text{Local } U(1)\text{symmetry} &\Rightarrow \text{Electromagnetic interaction,} \\ \text{Local } U(n)\text{symmetries} &\Rightarrow \text{non-Abelian gauge interactions.} \end{aligned}$$

To illustrate this principle, we will now "rederive" Quantum Electrodynamics from the requirement of local $U(1)$ symmetry. We would like to construct a theory which is invariant under

$$\psi(t,\vec{x}) \longrightarrow e^{i\alpha(x)}\psi(t,\vec{x}), \tag{8}$$

for general $\alpha(x)$, also called a gauge transformation. An immediate consequence of (8) is that the ordinary derivative loses its physical meaning. Consider the derivative along some direction n^μ :

$$n^\mu \partial_\mu \psi = \lim_{\epsilon \to 0} \frac{\psi(x+\epsilon n) - \psi(x)}{\epsilon}. \tag{9}$$

If we can rotate $\psi(x+\epsilon n)$ and $\psi(x)$ independently, (9) does not have a definite meaning, as can be seen from the last term in (7). That is, it does not make sense to compare the value of $\psi(x)$ at different points. So, to write down a sensible theory including kinetic terms for ψ, we need to introduce a new derivative, D_μ, such that:

$$D_\mu \psi(x) \longrightarrow e^{i\alpha(x)} D_\mu \psi(x). \tag{10}$$

To do this, assume we have an object $U(y,x)$ that transforms under (8) as

$$U(y,x) = e^{i\alpha(y)} U(y,x) e^{-i\alpha(x)}. \tag{11}$$

$U(y,x)$ "transports" the gauge transformation from $x \longrightarrow y$.

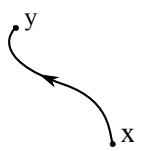

Figure 2: The parallel transport $U(y,x)$ transports the gauge transformation from x to y.

That is,

$$U(y,x)\psi(x) \longrightarrow e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)}e^{i\alpha(x)}\psi(x) = e^{i\alpha(y)}(U(y,x)\psi(x)), \tag{12}$$

transforming as $\psi(y)$. Since $\psi(y)$ and $U(y,x)\psi(x)$ have the same transformation properties, $\psi(y) - U(y,x)\psi(x)$ is well-defined.

Now take $y = x + \epsilon n$, and define

$$n^\mu D_\mu \psi = \lim_{\epsilon \to 0} \frac{\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)}{\epsilon}. \tag{13}$$

By construction,

$$D_\mu \psi \longrightarrow \lim_{\epsilon \to 0} e^{i\alpha(x+\epsilon n)} D_\mu \psi = e^{i\alpha(x)} D_\mu \psi \tag{14}$$

and

$$\mathscr{L} = -i\overline{\psi}(\gamma^\mu D_\mu - m)\psi \tag{15}$$

is invariant under (8). We now want to construct $U(y,x)$ explicitly. Since only local phase multiplication is a symmetry, $U(y,x)$ should be a phase, as we don't want to change other properties of $\psi(x)$. We begin infinitesimally:

$$U(x+\epsilon n, x) = 1 + i\epsilon n^\mu e A_\mu(x) + \ldots, \tag{16}$$

where e is a constant and $A_\mu(x)$ is a real vector field. Under the transformation (8),

$$U(x+\epsilon n, x) \longrightarrow e^{i\alpha(x+\epsilon n)} U(x+\epsilon n, x) e^{-i\alpha(x)}, \tag{17}$$

so that

$$1 + ie\epsilon n^\mu A_\mu(x) \longrightarrow e^{i\alpha(x)}(1 + i\epsilon n^\mu \partial_\mu \alpha(x))(1 + ie\epsilon n^\mu A_\mu(x)) e^{-i\alpha(x)}, \tag{18}$$

and hence

$$A_\mu(x) \longrightarrow A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x). \tag{19}$$

Finally, we have for the covariant derivative D_μ:

$$D_\mu \psi = \partial_\mu \psi - ieA_\mu \psi = (\partial_\mu - ieA_\mu)\psi. \tag{20}$$

Inserting the transformation laws for $A_\mu(x)$ and $\psi(x)$: (19) and (8), respectively, we have that $D_\mu \psi(x)$ transforms as $\psi(x)$. We want $A_\mu(x)$ to be a dynamical field, and hence we require a kinetic term for this vector field, which should be invariant under (19). To construct this, we note that $D_\mu(D_\nu \psi)$ transforms as ψ, and so does $(D_\mu D_\nu - D_\nu D_\mu)\psi$, so we define

$$[D_\mu, D_\nu] = [\partial_\mu - ieA_\mu, \partial_v - ieA_\nu] \equiv -ieF_{\mu\nu}, \tag{21}$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{22}$$

and we have that $F_{\mu\nu}\psi$ transforms as ψ, so that $F_{\mu\nu}$ is invariant.

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Lecture 3

We begin with some comments concerning gauge-symmetric theories:

1. A $U(1)$ local symmetry leads to the field $A_\mu(x)$, mediating interactions between the charge fields $\psi_i(x)$.

2. No mass term is allowed for A_μ or the gauge symmetry is broken.

3. It is possible to construct theories using other gauge invariant terms, for example
$$\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho},\ (F_{\mu\nu}F^{\mu\nu})^2,\ \bar{\psi}F_{\mu\nu}\gamma^\mu\gamma^\nu\psi. \tag{1}$$
All of these terms can be written as a total derivative, or are non-renormalizable.

4. Some of the terminology and constructs associated with gauge theories includes:
$$\begin{aligned} U(y,x) &: \text{Parallel transport},\\ A_\mu(x) &: \text{Connection},\\ F_{\mu\nu}(x) &: \text{Curvature},\\ D_\mu(x) &: \text{Covariant derivative}. \end{aligned}$$
These objects form the mathematical framework of the **fibre bundle**.

5. Gauge symmetry is not really a symmetry. The phase of ψ (and part of A_μ) does not carry physical information; there is a redundancy in our variables.

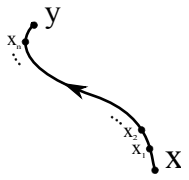

Figure 1: Dividing the path γ into infinitesimal segments.

6. Finally, we consider the explicit form of the finite parallel transport, $U(y,x)$: choose a path γ from $x \longrightarrow y$. We split the path into infinitesimal segments:
$$U_\gamma(y,x) = U(y,x_n)U(x_n,x_{n-1})\dots U(x_1,x), \tag{2}$$
where, from our result from the previous lecture for the infinitesimal parallel transport,
$$U(x_1,x) \approx \exp\left[ieA_\mu(x_1-x)^\mu\right], \tag{3}$$
and hence
$$U_\gamma(y,x) = \exp\left[ie\int_\gamma A_\mu(x)dx^\mu\right]. \tag{4}$$
Note that $U_\gamma(y,x)$ is not necessarily path-independent: in general, $U_{\gamma_1}(y,x) \neq U_{\gamma_2}(y,x)$. Let
$$U_\Gamma(x,x) = U_{-\gamma_2}(y,x)U_{\gamma_1}(y,x) \tag{5}$$
be the parallel transport associated with the closed loop shown in figure 2. By (4),
$$U_\Gamma(x,x) = \exp\left[ie\oint_\Gamma A_\mu dx^\mu\right],\ (\Gamma=\gamma_1-\gamma_2). \tag{6}$$
Using Stoke's theorem, we will see in the problem set that
$$U_\Gamma(x,x) = \exp\left[ie\int_\Sigma F_{\mu\nu}dx^\mu dx^\nu\right]. \tag{7}$$
$U_\Gamma(x)$ is the Wilson loop, and it is associated with phenomena such as the Aharonov-Bohm effect, and Berry's phase.

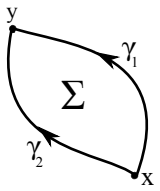

Figure 2: Paths γ_1 and γ_2 from x to y, and the enclosed area Σ.

1.3: NON-ABELIAN GENERALIZATIONS: YANG-MILLS THEORY

Now consider $\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$ and search for a theory invariant under

$$\Psi(x) \longrightarrow \Psi'(x) = V(x)\Psi(x), \tag{8}$$

or, with indices restored,

$$\Psi_i(x) \longrightarrow \Psi'_i(x) = V_i{}^j(x)\Psi_j(x). \tag{9}$$

Here, $V(x)$ is an $n \times n$ unitary matrix of unit determinant, that is, $V(x) \in SU(n)$. Let us construct the non-Abelian generalizations of the objects we studied in the Abelian case.

A. Covariant derivative:

Introduce $U(y,x) \in SU(n)$, transforming as

$$U(y,x) \longrightarrow V(y)U(y,x)V^\dagger(x). \tag{10}$$

Again, for $y^\mu = x^\mu + \epsilon n^\mu$, taking the limit $\epsilon \longrightarrow 0$, we expand $U(x+\epsilon n, x)$:

$$U(x+\epsilon n, x) = 1 + ig\epsilon n^\mu A_\mu(x) + \ldots, \tag{11}$$

where g is a constant, and $A_\mu(x)$ is an $n \times n$ matrix. As $U(y,x) \in SU(n)$, $A_\mu(x)$ is also necessarily traceless and hermitian; $A_\mu(x) = A^\dagger_\mu(x)$. Inserting this expansion into the transformation law (10), we obtain the gauge transformation law for the connection:

$$A_\mu(x) \longrightarrow V(x)A_\mu(x)V^\dagger(x) - \frac{i}{g}(\partial_\mu V(x))V^\dagger(x). \tag{12}$$

As before, we define the covariant derivative by

$$n^\mu D_\mu \Psi \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\Psi(x+\epsilon n) - U(x+\epsilon n, x)\Psi(x)\right]. \tag{13}$$

Hence, we have, with indices written explicitly,

$$(D_\mu \Psi)_i = \partial_\mu \Psi_i - ig(A_\mu)_i^j \Psi_j. \tag{14}$$

From (8) and (12), we have that, under a gauge transformation, $D_\mu\Psi(x) \longrightarrow V(x)(D_\mu\Psi)(x)$, and so, the Lagrangian $\mathscr{L} = -i\overline{\Psi}(\gamma^\mu D_\mu - m)\Psi$ is invariant.

B. Kinetic term for A_μ :

We note that under a gauge tranformation,

$$[D_\mu, D_\nu]\,\Psi \longrightarrow V\,[D_\mu, D_\nu]\,\Psi, \tag{15}$$

and that

$$\begin{aligned} [D_\mu, D_\nu] &= [\partial_\mu - igA_\mu, \partial_\nu - igA_\nu] \\ &= -ig(\partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu]) \\ &\equiv -igF_{\mu\nu}, \end{aligned} \tag{16}$$

with $F_{\mu\nu}$ an $n \times n$ matrix satisfying $F_{\mu\nu} = F_{\mu\nu}^\dagger$ and $\mathrm{Tr}F_{\mu\nu} = 0$. Under a gauge transformation, from (8), we have that

$$F_{\mu\nu}\Psi \longrightarrow F'_{\mu\nu}\Psi' = VF_{\mu\nu}\Psi, \tag{17}$$

and, as $\Psi' = V\Psi$, we have that

$$F'_{\mu\nu}(x) = V(x)F_{\mu\nu}(x)V^\dagger(x), \tag{18}$$

and so $F_{\mu\nu}$ is gauge covariant, as can be checked directly from (12). Hence, $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$ is invariant under gauge transformations.

C. The Lagrangian:

We can now write down an invariant $\mathscr{L}$:

$$\mathscr{L} = -\frac{c}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) - i\bar{\Psi}(\gamma^\mu D_\mu - m)\Psi, \tag{19}$$

with $\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$, $D_\mu = \partial_\mu - igA_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]$, and $A_\mu^\dagger = A_\mu$, $\mathrm{Tr}A_\mu = 0$. Explicitly, this Lagrangian is invariant under the local gauge transformation:

$$\begin{aligned}
\Psi(x) &\longrightarrow \Psi'(x) = V(x)\Psi(x), \\
A_\mu(x) &\longrightarrow A'_\mu(x) = V(x)A_\mu(x)V^\dagger(x) - \frac{i}{g}(\partial_\mu V(x))V^\dagger(x), \\
F_{\mu\nu}(x) &\longrightarrow F'_{\mu\nu}(x) = V(x)F_{\mu\nu}(x)V^\dagger(x),
\end{aligned}$$

where $V(x) = \exp\left[i\Lambda^a(x)T_a\right]$, and T_a are the generators of the Lie algebra, satisfying $[T_a, T_b] = if_{abc}T_c$.

Remarks:

1. A_μ is massless; introducing a mass term breaks gauge invariance.

2. It is convenient to expand A_μ as $A_\mu = A_\mu^a T_a$, with $a = 1, 2, \cdots n^2 - 1$. Here, A_μ is an $n \times n$ matrix, and A_μ^a are $n^2 - 1$ ordinary functions of x. Similarly, $F_{\mu\nu} = F_{\mu\nu}^a T_a$. We have that

$$F_{\mu\nu}^a T_a = \partial_\mu A_\nu^a T_a - \partial_\nu A_\mu^a T_a - ig\left[A_\mu^b T_b, A_\nu^c T_c\right], \tag{20}$$

   and so, explicitly,

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf_{abc}A_\mu^b A_\nu^c. \tag{21}$$

3. There is another gauge invariant term which can be constructed out of F at the quadratic level:

$$\epsilon_{\mu\nu\lambda\rho}\mathrm{Tr}(F^{\mu\nu}F^{\lambda\rho}). \tag{22}$$

   However, this term breaks CP-symmetry, and is also a total derivative. Nevertheless, this term is important at a non-perturbative level, as we will see in 8.325.

4. Non-Abelian gauge fields are associated with fibre bundles.

1.3.2: The Wilson loop

$$\begin{aligned}
U_\gamma(y, x) &= \lim_{n\to\infty} U(y, x_n)U(x_n, x_{n-1})\ldots U(x_1, x) \\
&= \lim_{\Delta x_j \to 0} \prod_{j=0}^{n} (1 + igA_\mu(x_j)\Delta x_j^\mu),
\end{aligned} \tag{23}$$

with $\Delta x_j^\mu = x_{j+1}^\mu - x_j^\mu$, $x_0 = x$, $x_{n+1} = y$. Note that the ordering is important in (23), as A_μ is a matrix, and so $[A_\mu(x_j), A_\nu(x_k)] \neq 0$ in general.

$$U_\gamma(y,x) = 1 + ig\sum_{j=0}^{n} A_\mu(x_j)\Delta x_j^\mu + (ig)^2 \sum_{j=0}^{n}\sum_{k=0}^{j-1} A_\mu(x_j)\Delta x_j^\mu A_v(x_k)\Delta x_k^v + \ldots + . \tag{24}$$

Now, we introduce $x^\mu(s)$ to parameterise γ:

$$x^\mu(0) = 0,\ x^\mu(1) = y^\mu,\ s \in [0,1]\,. \tag{25}$$

Then

$$\begin{aligned} U_\gamma(y,x) =& 1 + ig\int_0^1 ds_1 A_\mu(x(s_1))\frac{dx^\mu}{ds_1} + (ig)^2\int_0^1 ds_1\int_0^{s_1} ds_2 A_\mu(x(s_1))\frac{dx^\mu}{ds_1}A_v(x(s_2))\frac{dx^v}{ds_2} + \ldots + & (26)\\ =& \sum_{n=0}^{\infty}(ig)^n\int_0^1 ds_1\int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n A_{\mu_1}(x(s_1))\frac{dx^{\mu_1}}{ds_1}\ldots A_{\mu_n}(x(s_n))\frac{dx^{\mu_n}}{ds_n} & (27)\\ \equiv& P\exp\left[ig\int_0^1 ds\frac{dx^\mu}{ds}A_\mu(x(s))\right] & (28)\\ =& P\exp\left[ig\int_\gamma A_\mu(x)dx^\mu\right]. & (29)\end{aligned}$$

By construction, under a gauge transformation,

$$U_\gamma(y,x) \longrightarrow V(y)U_\gamma(y,x)V^\dagger(x). \tag{30}$$

To prove this directly using (12) is slightly non-trivial. As in the Abelian case, in general $U_{\gamma_1}(y,x) \neq U_{\gamma_2}(y,x)$. For a closed loop Γ, $U_\Gamma(x,x)$ is nontrivial. The non-Abelian generalisation of Stokes' theorem can be used to relate the parallel transport around the loop to the flux passing through the loop. For an infinitesimal loop,

$$U_\Gamma(x,x)\Psi - \Psi = \frac{1}{2}F_{\mu\nu}\sigma^{\mu\nu}\Psi, \tag{31}$$

where $\sigma^{\mu\nu}$ is the area element encircled by the loop. Under a gauge transformation,

$$U_\Gamma(x,x) \longrightarrow V(x)U_\Gamma(x,x)V^\dagger(x), \tag{32}$$

and hence,

$$W_\Gamma(x) = \mathrm{Tr}(U_\Gamma(x,x)) = \mathrm{Tr}(P\exp ig\oint_\Gamma A_\mu dx^\mu) \tag{33}$$

is gauge invariant. This is the non-Abelian Wilson loop. It is a very important object.

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Lecture 4

1.3.3: Field equations and conservation laws

We begin with the Lagrangian we considered in the last lecture:

$$\begin{aligned}\mathscr{L} &= -\frac{c}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) - i\bar{\Psi}(\gamma^\mu D_\mu - m)\Psi \\ &= -\frac{c}{4}\mathrm{Tr}(F^a_{\mu\nu}F^{\mu\nu b}T^aT^b) - i\bar{\Psi}(\gamma^\mu(\partial_\mu - igA^a_\mu T^a) - m)\Psi. \end{aligned} \tag{1}$$

For the gauge group $SU(n)$, we can choose T^a such that $\mathrm{Tr}(T^aT^b) = \delta_{ab}$, and so $c = 1$. We now obtain the equations of motion.

A. Varying A_μ :

$$\delta F^a_{\mu\nu} = \partial_\mu \delta A^a_\nu - \partial_\nu \delta A^a_\mu + g f^a_{bc}(\delta A^b_\mu A^c_\nu + A^b_\mu \delta A^c_\nu) = (D_\mu \delta A_\nu)^a - (D_v \delta A_\mu)^a \tag{2}$$

where $(D_\mu \delta A_\nu)^a \equiv \partial_\mu \delta A^a_\nu + g f^a_{bc} A^b_\mu \delta A^c_\nu$. Now,

$$\delta\mathscr{L} = -\frac{1}{2}\delta F^a_{\mu\nu}F^{\mu\nu a} - g\bar{\Psi}\gamma^\mu T^a \Psi \delta A^a_\mu, \tag{3}$$

and so

$$(D_\mu F^{\mu\nu})^a = J^{\nu a}, \tag{4}$$

where $(D_\mu F^{\mu\nu})^a \equiv \partial_\mu F^{\mu\nu a} + g f^a_{bc} A^b_\mu F^{\mu\nu c}$, and $J^{\nu a} \equiv g\bar{\Psi}\gamma^\nu T^a \Psi$. Equation (4) can also be written as:

$$\partial_\mu F^{\mu\nu a} = j^{\nu a} \tag{5}$$

with $j^{\nu a} \equiv g\bar{\Psi}\gamma^\nu T^a\Psi - g f^a_{bc} A^b_\mu F^{\mu\nu c}$.

B. Varying Ψ :

$$(\gamma^\mu D_\mu - m)\Psi = 0. \tag{6}$$

We note for emphasis that this is a matrix equation. Now, we recall that in quantum electrodynamics, $\partial_\mu F^{\mu\nu} = j^\nu$, with $j^\nu \equiv e\bar{\Psi}\gamma^\nu\Psi$ and $\partial_\nu j^\nu = 0$, that is, j^ν is a conserved current. When $j = 0$, A_μ is a free field, and we obtain free electromagnetic wave solutions.

Remarks:

1. In the non-Abelian case, the theory for A_μ remains interacting with $\Psi = 0$. In quantum electrodynamics, A_μ is neutral, whereas, in the non-Abelian case, A^a_μ carries the group index, and so is charged under itself, leading to self-interaction.

2. In terms of $F_{\mu\nu} = F^a_{\mu\nu}T^a$ and $J^\nu = J^{\nu a}T^a$, we have from (4) that

$$D_\mu F^{\mu\nu} = J^\nu, \tag{7}$$

with $D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} - ig\,[A_\mu, F^{\mu\nu}]$, and under a gauge transformation,

$$\begin{aligned} F_{\mu\nu} &\longrightarrow V F_{\mu\nu} V^\dagger, \\ D_\mu F^{\mu\nu} &\longrightarrow V D_\mu F^{\mu\nu} V^\dagger. \end{aligned} \tag{8}$$

more generally, for any $X = X^aT^a$, which transforms as $X \longrightarrow VXV^\dagger$, $D_\mu X = \partial_\mu X - ig\,[A_\mu, X]$ transforms as $D_\mu X \longrightarrow V(D_\mu X)V^\dagger$. From (7), we have that

$$J^\nu \longrightarrow V J^\nu V^\dagger. \tag{9}$$

This will be checked directly in the problem sets.

3. Acting with D_ν on (7),

$$\begin{aligned} D_\nu D_\mu F^{\mu\nu} =& \frac{1}{2}\left[D_\nu, D_\mu\right] F^{\mu\nu} \\ =& \frac{1}{2}\left[F_{\nu\mu}, F^{\mu\nu}\right] = 0, \end{aligned}$$

and so $D_\nu J^\nu = 0$. This can also be checked directly from the equations of motion.

4. J^ν is precisely the conserved Noether current for global $SU(n)$ symmetry in the absence of A_μ. For $A_\mu \neq 0$, J^ν is covariantly conserved. Of course, (1) is also invariant under global transformations

$$\Psi(x) \longrightarrow V\Psi(x),\ A_\mu(x) \longrightarrow V A_\mu(x) V^\dagger, \tag{10}$$

with V a position-independent $SU(n)$ matrix. The Noether current for this global symmetry is precisely j^ν, which was introduced in (5). From (5),

$$\partial_\mu j^\mu = 0. \tag{11}$$

Note that j^ν depends on A_μ non-trivially, which is true if and only if A^a_μ is charged. j^ν is not gauge invariant: it does not have good transformation properties.

1.3.4 Further generalizations

A **representation of a Lie group** G on a vector space V is a linear action

$$g \cdot v \in V \tag{12}$$

for $g \in G$, $v \in V$, such that

$$g_1 \cdot (g_2 \cdot v) = (g_1 \circ g_2) \cdot v \tag{13}$$

where $\circ$ is the group product. Here, V is the **representation space**.

A **representation of a Lie algebra** $\mathfrak{g}$ on a vector space V is a linear action

$$x \cdot v \in V \tag{14}$$

for $g \in G$, $v \in V$, such that

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v) \tag{15}$$

where $[\,,\,]$ is defined using the group product. The concept of a representation is, again, tailor-made for physics. Here, V is the physical space, and the same abstract group can appear in different physical contexts, with different V (with different representations). We note that a representation for G induces a representation for $\mathfrak{g}$, and visa versa.

Example 1: Angular momentum, $SU(2)$

A spin-j representation is described by $(2j+1) \times (2j+1)$ matrices, acting on a $(2j+1)$-dimensional V. For $SU(n)$, representing the group as $n \times n$ unitary matrices acting on n-dimensional complex vectors gives the **fundamental representation.** Here,

$$A_\mu = A^a_\mu T^a \in \mathfrak{g}, \tag{16}$$

with T^a in the fundamental representation. More generally, a representation r of $\mathfrak{g}$ (or G) of dimension d_r is defined by $d_r \times d_r$ matrices $T^{(r)}_a$ that represent the generators, satisfying

$$\left[T^{(r)}_a, T^{(r)}_b\right] = i f^c_{ab} T^{(r)}_c. \tag{17}$$

One can prove that, for compact groups,

$$\mathrm{Tr}(T^{(r)}_a T^{(r)}_b) = C(r)\delta_{ab}, \tag{18}$$

where $C(r)$ is a positive number depending on the representation r. For non-compact groups, $\mathrm{Tr}(T^{(r)}_a T^{(r)}_b)$ is not positive-definite. Amongst all representations, there is a special one for all G (and $\mathfrak{g}$). The **adjoint representation**,

which is universal for all Lie groups and Lie algebras, has as its representation space the vector space for the Lie algebra itself. That is, $V = \mathfrak{g}$ ($\dim V = \dim \mathfrak{g}$). The action of the Lie algebra is defined, for $x \in V = \mathfrak{g}$, $y \in \mathfrak{g}$, by

$$y \cdot x \equiv [y, x]\,. \tag{19}$$

We need to show that this rule satisfies (15), that is,

$$y_1 \cdot (y_2 \cdot x) - y_2 \cdot (y_1 \cdot x) = [y_1, y_2] \cdot x. \tag{20}$$

The left-hand side of this equation is $[y_1, [y_2, x]] - [y_2, [y_1, x]]$, and the right-hand side is $[[y_1, y_2]\,, x]$. The equality of these follows from the Jacobi identity, which is an automatic consequence of the associativity of the group product, so (19) indeed gives a representation. For $SU(n)$, the $T_a^{(adj)}$ are $(n^2-1)\times(n^2-1)$ matrices. Now we go back to the theory, and generalize as follows: we consider a general compact group G in place of $SU(n)$, with Ψ in the vector space of some representation r of G. $A_\mu = A_\mu^a T_a^{(r)} \in \mathfrak{g}$. Now we take

$$\mathscr{L} = -\frac{c}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) - i\bar{\Psi}(\gamma^\mu D_\mu - m)\Psi, \tag{21}$$

with $F_{\mu\nu} \equiv F_{\mu\nu}^a T_a^{(r)}$, $D_\mu \equiv \partial_\mu - igA_\mu^a T_a^{(r)}$. Since $\mathrm{Tr}(T_a^{(r)} T_b^{(r)}) = C(r)\delta_{ab}$, to maintain canonical normalization, $c = \frac{1}{C(r)}$. Here, compactness of G is essential, as non-compactness leads to the wrong sign for kinetic terms of A_μ^a. Now, the action of the Lie group G on $V = \mathfrak{g}$ is given, for $g \in G$, $x \in \mathfrak{g}$, by

$$g \cdot x \equiv g \circ x \circ g^{-1}. \tag{22}$$

It is easy to check that this satisfies the group action product rule (13). For infinitesimal $g = 1 + iy$, $y \in \mathfrak{g}$, this action reduces to (19). In matrix form, for $x \in V = \mathfrak{g}$, $x = x^b T_b$ (the upper and lower indices can be distinct for this general treatment)

$$T_a^{(adj)} \cdot x = [T_a, x] = [T_a, T_b]\, x^b = i f_{ab}^c T_c x^b. \tag{23}$$

On the other hand,

$$T_a^{(adj)} \cdot x = (T_a^{(adj)} \cdot x)^c T_c \tag{24}$$

where $(T_a^{(adj)} x)^c \equiv (T_a^{(adj)})_b^c x^b$. Hence,

$$(T_a^{(adj)})_b^c = i f_{ab}^c. \tag{25}$$

Remarks:

1. Consider M^a in the adjoint representation, $a = 1, \ldots, \dim \mathfrak{g}$;

$$\begin{aligned}(D_\mu M)^a =& \partial_\mu M^a - igA_\mu^a (T_b^{(adj)} \cdot M)^a \\ =& \partial_\mu M^a + g f_{bc}^a A_\mu^b M^c,\end{aligned}$$

where the second line follows from (25). In matrix form, with $M = M^a T_a^{(r)}$ for any representation r,

$$\begin{aligned}D_\mu M =& \partial_\mu M - igA_\mu^a T_a^{(adj)} \cdot M \\ =& \partial_\mu M - igA_\mu^a T^{(r)} \\ =& \partial_\mu M - ig\,[A_\mu, M]\end{aligned}$$

with $A_\mu = A_\mu^a T_a$.

2. Under a gauge transformation,

$$\begin{aligned}M \longrightarrow& V M V^{-1}, \\ M^a T_a \longrightarrow& M^a V T_a V^{-1},\end{aligned}$$

and hence,

$$M^a \longrightarrow M^a D_a^b, \tag{26}$$

where $V T_a V^{-1} \equiv D_a^b T_b$.

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Lecture 5

1.4: QUANTIZATION OF NON-ABELIAN GAUGE THEORIES

1.4.1: Gauge Symmetries

Gauge symmetry is not a true symmetry, but a reflection of the fact that a theory possesses redundant degrees of freedom. A gauge symmetry implies the existence of different field configurations which are equivalent. For example, in the case of $U(1)$,

$$\psi \longrightarrow e^{i\alpha(x)}\psi,$$
$$A_\mu \longrightarrow A_\mu + \frac{1}{e}\partial_\mu \alpha(x),$$

the phase of ψ is not a physical degree of freedom. Similarly, nor is the longitudinal part of A_μ. A massless spin-1 representation of the Lorentz group has only two polarizations. A_μ has four components. Thus to have a Lorentz covariant formulation, we require gauge symmetries to get rid of the extra degrees of freedom.

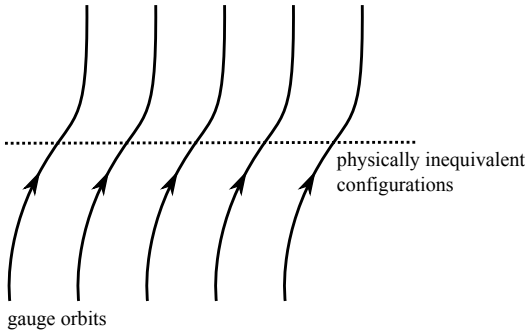

Figure 1: Equivalent gauge orbits in configuration space.

When quantizing the theory, we should separate the redundant and physical degrees of freedom. We need to make sure only physical modes contribute to observables. This leads to complications in dealing with gauge theories. There are two general approaches:

1. Isolate the physical degrees of freedom: fix a gauge and quantize the resulting constrained system. This method is used, for example, in axial gauge quantization in quantum electrodynamics.

2. Retain the unphysical modes, or even introduce additional modes, but make sure that they do not contribute to any physical observables. This method is used, for example, in covariant path integral quantization.

For the first complication in the path integral quantization, consider, for example, the path integral for a scalar field

$$\int \mathfrak{D}\phi\, e^{-\int d^d x \frac{1}{2}\phi^T K \phi + V(\phi) - J^T \phi} = e^{-V(\frac{\delta}{\delta J})} e^{\frac{1}{2}\int d^4 x J^T K^{-1} J}, \tag{1}$$

where K is the kinetic operator $(-\partial^2 + m^2)$ and K^{-1} is the propagator for ϕ. For gauge theories, the inverse of K is not defined. For example, in quantum electrodynamics,

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= A_\mu K^{\mu\nu} A_\nu + \text{total derivatives}, \end{aligned}$$

with $K^{\mu\nu} = \partial^2\eta^{\mu\nu} - \partial^\mu\partial^\nu$. We see that $K^{\mu\nu}\partial_\nu\alpha(x) = 0$ for any $\alpha(x)$, and so the matrix is singular. These zero eigenmodes are the configuarations which are gauge-equivalent to 0. Non-Abelian gauge theories have the same quadratic kinetic terms as quantum electrodynamics. In order for K to have an inverse, we need to separate gauge orbits with physically inequivalent configurations.

1.4.2 Fadeev-Popov method:

Example 1: A trivial example

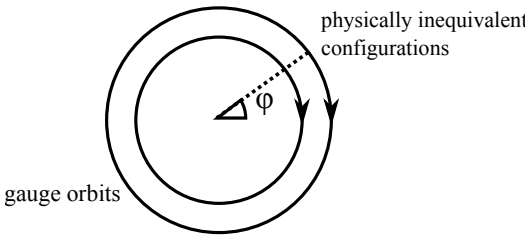

Figure 2: The radial direction gives inequivalent configurations, and the circles of fixed radius are the gauge orbits.

Consider

$$W = \int dxdy\, e^{f(x,y)} \tag{2}$$

and suppose $f(x,y)$ only depends on $r = \sqrt{x^2+y^2}$. Then

$$\begin{aligned} W &= \int drd\phi\, re^{f(r)} \\ &= 2\pi \int dr\, re^{f(r)}, \end{aligned}$$

where the 2π is the factorized orbit volume. Equivalently, we can insert a delta function. More explicitly, we can insert a factor of $\int d\phi_0 \delta(\phi - \phi_0) = 1$. Then

$$W = \int d\phi_0 \int dxdy e^{f(x,y)} \delta(\phi - \phi_0). \tag{3}$$

The $\int d\phi_0$ integrates over the gauge orbit, and the other factor integrates over a section of non-gauge equivalent configurations.

Example 2: Gauge theories

We consider pure-gauge theories only; adding matter fields is trivial.

$$Z = \int \prod_{a=1}^{\dim G} \mathfrak{D}A^a_\mu(x)\, e^{iS[A_\mu]}. \tag{4}$$

We define a set of gauge-fixing conditions:

$$f_a(A) = 0,\ a = 1, \ldots, \dim G, \tag{5}$$

in order to select a section of non-equivalent configurations:

$$1 = \int \prod_a d\Lambda_a(x) \delta(f_a(A_\Lambda)) \det\left[\frac{\delta f_a(A_\Lambda(x))}{\delta\Lambda_b(y)}\right]. \tag{6}$$

Here, the determinant is the determinant of both the color and function space. Inserting (6) into (4), using an abridged notation,

$$Z = \int d\Lambda \int \mathfrak{D}A\, e^{iS[A]} \delta(f(A_\Lambda)) \det\left[\frac{\delta f(A_\Lambda)}{\delta\Lambda}\right]. \tag{7}$$

Now we observe that $\mathfrak{D}A = \mathfrak{D}A_\Lambda$, as gauge transformations correspond to unitary transformations plus shifts, and that $S[A] = S[A_\Lambda]$, because of the defining gauge symmetry. Hence,

$$\begin{aligned} Z &= \int d\Lambda \int \mathfrak{D}A_\Lambda e^{iS[A_\Lambda]} \delta(f(A_\Lambda)) \det\left[\frac{\delta f(A_\Lambda)}{\delta\Lambda}\right] \\ &= \int d\Lambda \int \mathfrak{D}A e^{iS[A]} \delta(f(A)) \det\left[\frac{\delta f(A)}{\delta\Lambda}\right], \end{aligned}$$

as A_Λ is a dummy integration variable. So, again we factor the partition function into the gauge volume $\int d\Lambda$ and a path-integral over gauge-inequivalent configurations which is independent of Λ. We redefine this latter factor to be the new partition function; that is,

$$Z \equiv \int \mathfrak{D}A e^{iS[A]} \delta(f(A)) \det\left[\frac{\delta f(A)}{\delta\Lambda}\right]. \tag{8}$$

Example 3: Axial gauge

$$f_a(A) = A^a_z = 0. \tag{9}$$

From this,

$$f_a(A_\Lambda) = A^a_z + \frac{1}{g}(\partial_z \Lambda_a + g f_{abc} A^b_z \Lambda^c), \tag{10}$$

and hence,

$$\left.\frac{\delta f_a(A_\Lambda(x))}{\delta\Lambda_b(x')}\right|_{\Lambda=0} = \frac{1}{g}\partial_z \delta_{ab}\delta(x-x'). \tag{11}$$

We see that the Jacobian is independent of A_μ: its determinant only gives an overall constant in the partition function. Hence,

$$Z = \int \mathfrak{D}A e^{iS[A]} \prod_a \delta(A^a_z) \tag{12}$$

up to a constant. This is a particularly simple form. However, the drawback is that Lorentz covariance has been broken. In a general covariant gauge, both the determinant and delta-function factors are more difficult to work with. We need to employ additional tricks.

(i) Determinant factor

Recall that

$$\int \prod_a d\Psi d\bar\Psi e^{\bar\psi_a M_{ab}\psi_b} = \det M_{ab}, \tag{13}$$

where the ψ_a and $\bar\psi_b$ are independent Grassman variables. Hence, the Fadeev-Popov determinant is given by

$$\det\left[\left.\frac{\delta f_a(A_\Lambda(x))}{\delta\Lambda_b(y)}\right|_{\Lambda=0}\right] = \int \mathfrak{D}C_a(x)\mathfrak{D}\bar C_a(x)\, e^{i\int d^4x d^4y\, \bar C_a(x) \left.\frac{\delta f_a(A_\Lambda(x))}{\delta\Lambda_b(y)}\right|_{\Lambda=0} C_b(y)}, \tag{14}$$

where $C_a(x)$ and $\bar C_a(x)$, $a = 1, \ldots, \dim G$, are real fermionic fields with no spinor indices. These are the ghost fields.

(ii) Delta-function factor

Again, the method is to write this factor in the form of an exponential. Firstly, generalize

$$\delta(f_a(A)) \longrightarrow \delta(f_a(A) - B_a(x)) \tag{15}$$

where $B_a(x)$ is an arbitrary function. This does not change the Fadeev-Popov determinant. Therefore, Z is independent of $B_a(x)$, and so we can weight the integrand of Z with a Gaussian distribution of $B_a(x)$. That is,

$$Z = \int \prod_a \mathfrak{D}B_a(x) e^{-i\int d^4x \frac{1}{2\xi}B^2_a(x)} \times \int \mathfrak{D}A e^{iS[A]} \delta(f(A)) \det\left[\left.\frac{\delta f(A)}{\delta\Lambda}\right|_{\Lambda=0}\right]. \tag{16}$$

Collecting (14) and (16), we find for Z,

$$Z = \int \mathfrak{D}A \mathfrak{D}C_a D\bar{C}_a \, e^{iS_{eff}[A,C,\bar{C}]} \tag{17}$$

with the effective action S_{eff} given by

$$S_{eff}[A] = S_{YM}[A] - \frac{1}{2\xi}\int d^4x \, f_a^2(A) + \int d^4x d^4y \, \bar{C}_a(x) \left[\frac{\delta f_a(A_\Lambda(x)}{\delta \Lambda_b(y)}\bigg|_{\Lambda=0} \right] C_b(y). \tag{18}$$

Example 4: Lorentz gauge

$$f_a(A_\mu) = \partial^\mu A_\mu^a. \tag{19}$$

We have that $(A_\mu^a)_\Lambda(x) = A_\mu^a(x) + \frac{1}{g}(\partial_\mu \Lambda_a(x) + g f^{acd} A_\mu^c \Lambda^d)$, and so the Jacobian is given by

$$\frac{\delta f_a(A_\Lambda(x))}{\delta \Lambda_b}\bigg|_{\Lambda=0} = \frac{1}{g}\left[\partial^\mu \delta_{ab}\partial_\mu + g^{cab}A_\mu^c\right]\delta^{(4)}(x-y), \tag{20}$$

giving

$$\mathscr{L}_{eff} = \mathscr{L}[A] + \mathscr{L}_{gf} + \mathscr{L}_{gh}, \tag{21}$$

with

$$\mathscr{L}_{gf} = -\frac{1}{2\xi}(\partial^\mu A_\mu^a), \quad \mathscr{L}_{gh} = \int d^4x \, \bar{C}_a(x)\partial^\mu D_\mu C_a(x), \tag{22}$$

where $D_\mu C_a(x) \equiv \partial_\mu C_a(x) + g f_{abd} A_\mu^b C_d$. From this, we can derive the Feynman rules for the theory.

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Lecture 6

1.5: BRST SYMMETRY, PHYSICAL STATES AND UNITARITY

1.5.1: Becchi-Rouet-Stora-Tyutin (BRST) Symmetry

From the last lecture, we have

$$Z = \int \mathfrak{D}A^a_\mu \mathfrak{D}C_a D\bar{C}_a \, e^{iS_{eff}[A^a_\mu, C, \bar{C}]}, \tag{1}$$

with

$$S_{eff}\left[A, C, \bar{C}\right] = S_0\left[A\right] - \frac{1}{2\xi}\int d^4x \, f_a^2(A) + \int d^4x d^4y \, \bar{C}_a(x) \left[\frac{\delta f_a(A_\Lambda(x)}{\delta \Lambda_b(y)}\bigg|_{\Lambda=0}\right] C_b(y), \tag{2}$$

where $f_a(A)$ is the gauge-fixing function and $S_0\left[A\right] = \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}$ is the pure Yang-Mills action. $S_0\left[A\right]$ is invariant under the gauge transformation

$$A^a_\mu \longrightarrow A^a_\mu + D_\mu \Lambda^a \tag{3}$$

where $D_\mu \Lambda^a = \partial_\mu \Lambda^a + f^{abc}A^b_\mu \Lambda^c$. We note that in (1) we integrate over all $A^a_\mu(x)$, including the unphysical configurations, but by construction Z should only receive contributions from the physical $A^a_\mu(x)$. We also note that $Z = \langle 0, +\infty | \, 0, -\infty \rangle$. $S_{eff}\left[A, C, \bar{C}\right]$ no longer has gauge symmetries, but it has a hidden global fermionic symmetry, the BRST symmetry, which is, in fact, a remnant of the gauge symmetry. To see this, it is convenient to introduce an auxillary field $h_a(x)$:

$$Z = \int \mathfrak{D}A^a_\mu \mathfrak{D}h_a \mathfrak{D}C_a D\bar{C}_a \, e^{iS_{eff}[A^a_\mu, C, \bar{C}, h]} \tag{4}$$

with

$$S_{eff}\left[A, C, \bar{C}, h\right] = S_0\left[A\right] + \frac{\xi}{2}\int d^4x \, h_a^2 + \int d^4x \, h_a(x) f_a(x) + \mathscr{L}_{gh}. \tag{5}$$

Now, consider the following (BRST) transformations:

$$\delta_B A^a_\mu = \eta (D_\mu C)^a \equiv \eta s(A^a_\mu) \tag{6}$$

$$\delta_B \bar{C}^a = -\eta h^a \equiv \eta s(\bar{C}^a) \tag{7}$$

$$\delta_B C^a = -\frac{1}{2} g\eta f^{abc} C^b C^c \equiv \eta s(C^a) \tag{8}$$

$$\delta_B h^a = 0 \equiv \eta s(h^a) \tag{9}$$

with η an anticommuting constant parameter. Then, in general,

$$\delta_B \phi \equiv \eta s(\phi), \quad \phi = A^a_\mu, C^a, \bar{C}^a, h^a. \tag{10}$$

$s(\phi)$ takes ϕ to a field of opposite 'fermionic parity'. We note some of the important properties of s :

i.

$$s(\phi_1\phi_2) = s(\phi_1)\phi_2 \pm \phi_1 s(\phi_2), \tag{11}$$

where the + sign is for ϕ_1 bosonic, and the − sign is for ϕ_1 fermionic.

ii.

$$s^2(\phi) = 0. \tag{12}$$

For example, $s^2(\bar{C}^a) = 0$ and $s^2(C^a) = 0$, which follows from the Jacobi identity.

iii. From (i) and (ii), we have that

$$s^2(F(\phi)) = 0. \tag{13}$$

iv. $s(A^a_\mu)$ is the same as the infinitesimal gauge transformation of A^a_μ with Λ^a replaced by C^a.

Based on the above properties, we will now prove that $\delta_B S = 0$.

We first show that

$$S = S_0 + \int d^4x\, s(F(x)) \tag{14}$$

with $F(x) = -\bar{C}_a f_a - \frac{\xi}{2}\bar{C}_a h_a$, so that

$$s(F(x)) = h_a f_a + \bar{C}_a s(f_a(A_\mu)) + \frac{\xi}{2} h_a^2. \tag{15}$$

This can be established by showing that

$$\int d^4x\, \bar{C}_a(x) s(f_a(A_\mu(x))) = \int d^4y d^4x\, \bar{C}_a(x) \left[\frac{\delta f_a(A_\Lambda(x)}{\delta \Lambda_b(y)}\bigg|_{\Lambda=0} \right] C_b(y), \tag{16}$$

which is left as an exercise to the reader. Then, we have that

$$\delta_B S = \delta_B S_0 + \eta \int d^4x\, s^2(F(x)), \tag{17}$$

and these terms are separately zero by the properties (iii) and (iv) shown above.

□

The BRST symmetry implies the existence of a conserved fermionic charge Q_B.

$$\delta_B \phi = i\,[\eta Q_B, \phi] = \eta s(\phi) \tag{18}$$

or, equivalently,

$$\begin{aligned} s(\phi) =& i\,[Q_B,\phi]_\pm \\ =& \begin{cases} i\,[Q_B,\phi]\,, & \phi \text{ bosonic,} \\ i\,\{Q_B,\phi\}\,, & \phi \text{ fermionic.} \end{cases} \end{aligned}$$

Since $s^2(\phi) = 0$, we have that

$$\left[Q_B, [Q_B,\phi]_\pm\right]_\mp = 0. \tag{19}$$

That is, $\left[Q_B^2, \phi\right] = 0$ for any ϕ, and hence,

$$Q_B^2 = 0. \tag{20}$$

We can also define a ghost number, which is conserved:

$$\mathrm{gh}\,[C] = 1,\ \mathrm{gh}[\bar{C}] = -1,\ \mathrm{gh}\,[Q_B] = 1,\ \mathrm{gh}[\tilde{\phi}] = 0 \tag{21}$$

for any other field $\tilde{\phi}$.

1.5.2: Physical States and Unitarity

Physical states should be independent of the gauge choice. $Z = \langle 0, +\infty |\, 0, -\infty \rangle$ is so by construction, as it should be independent of $f_a(A)$. We now consider more general observables. More generally, we should have that

$$0 = \delta_g \langle f |\, i \rangle = i \langle f |\, \delta_g S\, | i \rangle \tag{22}$$

where δ_g represents the change under the variation of the gauge-fixing condition $f_a(A)$. Note from (14) we have that

$$\begin{aligned}\delta_g S &= \int d^4x\, s(\delta_g F(x)) \\ &= -\int d^4x\, s(\bar{C}_a \delta f_a) \\ &= -i\int d^4x\, \left\{Q_B, \bar{C}_a \delta f_a(A)\right\},\end{aligned}$$

and so it must be true that

$$\int d^4x \,\langle f|\left\{Q_B, \bar{C}_a \delta f_a(A)\right\}|i\rangle = 0 \tag{23}$$

for arbitrary $\delta f_a(A)$ for a physical observable, and so

$$Q_B\,|i\rangle = Q_B\,|f\rangle = 0. \tag{24}$$

That is, a physical state $|\psi\rangle$ should satisfy

$$Q_B\,|\psi\rangle = 0. \tag{25}$$

Similarly, by considering

$$\delta_g \langle f|\, O_1 \dots O_n\,|i\rangle = 0, \tag{26}$$

we find that

$$[Q_B, O] = 0 \tag{27}$$

and so O should be gauge invariant (if it does not contain ghost fields). Note that any state of the form

$$|\psi\rangle = Q_B\,|\dots\rangle \tag{28}$$

satisfies $Q_B\,|\psi\rangle = 0$, but that in this case, $\langle\chi|\,\psi\rangle = 0$ for any physical state $|\chi\rangle$. Such a state $|\psi\rangle$ is called a null state. All physical observables involving a null state vanish. If $|\psi'\rangle$, $|\psi\rangle$ satisfying (25) are related by

$$|\psi'\rangle = |\psi\rangle + Q_B\,|\dots\rangle, \tag{29}$$

they will have the same inner product with all physical states, and thus are equivalent. We introduce

$$\begin{aligned}\mathscr{H}_{closed} &= \{|\psi\rangle :\, Q_B\,|\psi\rangle = 0\}, \\ \mathscr{H}_{exact} &= \{|\psi\rangle :\, |\psi\rangle = Q_B\,|\dots\rangle\}, \\ \mathscr{H}_{phys} &= \frac{\mathscr{H}_{closed}}{\mathscr{H}_{exact}}.\end{aligned}$$

That is, $\mathscr{H}_{phys}$ is the cohomology of Q_B. In summary:

1. Defining $\mathscr{H}_{big}$ to be the Fock space composed from $A_\mu, C, \bar{C}$, we have that

$$\mathscr{H}_{phys} \subset \mathscr{H}_{closed} \subset \mathscr{H}_{big}. \tag{30}$$

2. By restricting to $\mathscr{H}_{phys}$ and gauge invariant O, $\langle f|\, O_1 \dots O_n\,|i\rangle$ does not depend on the gauge choice.

3. Our path integral construction guarantees that only physical states contribute in the intermediate state.

Example 1: Quantum electrodynamics in Feynman gauge ($\xi = 1$)

$$\begin{aligned}\mathscr{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \partial_\mu C \partial^\mu \bar{C} \\ &= -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \partial_\mu C \partial^\mu \bar{C}\end{aligned}$$

Under the BRST transformation:

$$\begin{aligned}\delta_B A_\mu =&\eta\partial_\mu C\\ \delta_B \bar{C} =&-\eta\partial_\mu A^\mu\\ \delta_B C =&0,\end{aligned}$$

and so

$$\begin{aligned}[Q_B, A_\mu] =&-i\partial_\mu C\\ \left[Q_B, \bar{C}\right] =&i\partial_\mu A^\mu\\ [Q_B, C] =&0.\end{aligned}$$

It is left as an exercise for the reader to find the explicit form for Q_B. Now, we set $\mathscr{H}_{big}$ to be the set of states formed by acting with creation operators for $A_\mu, \bar{C}, C$ on the ground state $|0\rangle$. Imposing $Q_B|\psi\rangle = 0$ gives us $\mathscr{H}_{closed}$ and $\mathscr{H}_{phys}$. For illustration, consider the one-particle state:

$$\mathscr{H}_{big} = \{|e_\mu, p\rangle, |c, p\rangle, |\bar{c}, p\rangle\}. \tag{31}$$

Then, $Q_B|e, p\rangle = Q_B e \cdot A|0\rangle = -e \cdot p|c, p\rangle$ from $[Q_B, A_\mu] = -i\partial_\mu C$, and we obtain the physical state condition:

$$e.p = 0 \tag{32}$$

For $e.p \neq 0$, we get$|c, p\rangle$ null states. $Q_B|\bar{c}, p\rangle \propto p_\mu A^\mu(p)|0\rangle \neq 0$, and so the $|\bar{c}, p\rangle$ are non-physical states, and $p_\mu A^\mu(p)|0\rangle = |e = p, p\rangle$ is a null state. So, we have that

$$\mathscr{H}_{phys} = \{|e, p\rangle : e.p = 0,\ e^\mu \sim e^\mu + p^\mu\}. \tag{33}$$

Take $p^\mu = \left(p^0, 0, 0, p^3\right)$, $p^2 = 0$. Then, $e.p = 0$ implies that $e^\mu = \left(p^0, e_1, e_2, p^3\right)$, and $e \sim e + p$ implies that $e^\mu = (0, e_1, e_2, 0)$, and so, only transverse components of A_μ generate physical states.

Remarks:

1. While $A_0|0\rangle$ creates negative-norm states, they do not lie in the physical state space (giving a positive-definite norm on the physical state space).

2. Ghosts $C, \bar{C}$ make sure that these negative norm states do not contribute in intermediate steps.

8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 7

2: GENERAL ASPECTS

Now that we have studied almost all known quantum field theories, we return to studying physical observables in these theories. We will start with the simplest object:

$$\langle 0| O(x)O(y) |0\rangle , \tag{1}$$

that is, the vacuum two-point function. It is also one of the most important objects. We will first describe its general structure based on symmetries, unitarity and analyticity, without the specifics of any particular theory. We will then consider an example in a specific theory.

2.1: FIELD AND MASS RENORMALIZATIONS

For illustration, consider a scalar field theory, with Lagrangian $\mathscr{L}(\phi)$ which is Lorentz invariant and translation invariant. We examine

$$\begin{aligned} G_F(x,y) &= \langle 0| T(\phi(x)\phi(y)) |0\rangle , \text{ the Feynman function,} \\ G_+(x,y) &= \langle 0| \phi(x)\phi(y) |0\rangle , \text{ the Wightman function, and} \\ G_R(x,y) &= \Theta(x^0-y^0) \langle 0| \phi(x)\phi(y) |0\rangle , \text{ the retarded Green's function.} \end{aligned}$$

Here, $\Theta(x^0)$ is the heaviside function. Translation invariance implies that $G_F(x,y) = G_F(x-y)$, and Lorentz invariance implies that $G_F(x-y) = G_F((x-y)^2)$. In momentum space, we have that

$$G_F(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip.(x-y)} G_F(p), \tag{2}$$

and by Lorentz invariance, $G_F(p)$ depends only on p^2. Recall, in the free theory,

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2, \tag{3}$$

and $G_+^{(0)}(p;m_0^2) = 2\pi\Theta(p^0)\delta(p^2+m_0^2)$. The delta function constrains the support of the Green's function to the mass-shell, and the heaviside function constrains the support to the positive-energy sheet.

$$G_F(p;m_0^2) = \frac{-i}{p^2+m_0^2-i\epsilon}. \tag{4}$$

Now, for a general interacting theory,

$$G_+(x,y) = \sum_n \langle 0| \phi(x) |n\rangle \langle n| \phi(y) |0\rangle , \tag{5}$$

where we have just inserted a complete set of states, and where the summation heuristically also represents an integration over momentum. Now,

$$\begin{aligned} \langle 0| \phi(x) |n\rangle =& \langle 0| e^{-i\hat{p}.x}\phi(0)e^{i\hat{p}.x} |n\rangle \\ =& e^{ip_n.x} \langle 0| \phi(0) |n\rangle , \end{aligned}$$

and so,

$$G_+(x-y) = \sum_n e^{ip_n.(x-y)} |\langle 0| \phi(0) |n\rangle|^2 , \tag{6}$$

or, more explicitly,

$$G_+(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip.(x-y)} (2\pi)^4 \sum_n \delta^{(4)}(p-p_n) \left|\langle 0|\, \phi(0)\, |n\rangle\right|^2 . \tag{7}$$

We define $(2\pi)\Theta(p^0)\rho(-p^2) = (2\pi)^4 \sum_n \delta^{(4)}(p-p_n) \left|\langle 0|\, \phi(0)\, |n\rangle\right|^2$, a Lorentz invariant scalar function of p, with $\rho(-p^2) = 0$ for $p^2 > 0$, as all $|n\rangle$ should have $E_n = p_n^0 > 0$, and so $p_n^2 < 0$. We thus find that

$$\begin{aligned} G_+(p) =& 2\pi\Theta(p^0)\rho(-p^2) \\ =& \int_0^\infty d\mu^2\, 2\pi\delta(p^2+\mu^2)\Theta(p^0)\rho(\mu^2), \end{aligned}$$

and so, for an interacting theory, we obtain the general result:

$$\begin{aligned} G_+(p) &= \int_0^\infty d\mu^2\, \rho(\mu^2) G_+^{(0)}(p;\mu^2) \\ G_+(x-y) &= \int_0^\infty d\mu^2\, \rho(\mu^2) G_+^{(0)}(x-y;\mu^2) \end{aligned}$$

where $\rho(\mu^2)$ is the spectral function. Now we consider the structure of $\rho(-p^2)$. We have that $(2\pi)\Theta(p^0)\rho(-p^2) = (2\pi)^4 \sum_n \delta^{(4)}(p-p_n) \left|\langle 0|\, \phi(0)\, |n\rangle\right|^2$, where $\sum_n$ is a sum over all physical states: that is, a sum over single-particle states and multi-particle states. Firstly,

$$\begin{aligned} \sum_{\text{single-particle states}} =& \sum_i \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_k} \\ =& \sum_i \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2+m_i^2), \end{aligned}$$

where i indexes the particle species, and $\omega_k = \sqrt{m_i^2 + \vec{k}^2}$. Other than the m_i^2, this sum is completely determined by Lorentz symmetry. Now, we consider the multi-particle states. For a given $\vec{k}$, a continuum of ω are allowed. For example, for a two-particle state at $\vec{k} = 0$,

Figure 1: A two particle-state with zero total momentum.

$$\omega = \sqrt{k_1^2 + m_1^2} + \sqrt{k_2^2 + m_2^2}, \tag{8}$$

ω forms a continuum starting at $m_1 + m_2$, and so, k^2 starts at $-(m_1+m_2)^2$. We thus find

$$(2\pi)\Theta(p^0)\rho(-p^2) = \sum_i \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2+m_i^2)\Theta(k^0)(2\pi)^4\delta^{(4)}(p-k) \left|\langle 0|\, \phi(0)\, |k,i\rangle\right|^2 + 2\pi\Theta(p^0)\sigma(-p^2) \tag{9}$$

where the second term is the contribution from multi-particle states. Hence, we have that

$$\rho(-p^2) = \sigma(-p^2) + \sum_i \delta(p^2+m_i^2) Z_i \tag{10}$$

with $Z_i = \left|\langle 0|\, \phi(0)\, |k,i\rangle\right|^2$ a number which is independent of k, since $k^2 = -m_i^2$. $\sigma(-p^2)$ is non-zero for $-p^2 \geq 4m_1^2$, where m_1 is the smallest single-particle mass. So, for the Feynman function, we have

$$G_F(p) = \sum_j \frac{-iZ_j}{p^2+m_j^2-i\epsilon} + \int_{4m_1^2}^\infty d\mu^2\, \sigma(\mu^2) \frac{-i}{p^2+\mu^2-i\epsilon}. \tag{11}$$

It is convenient to introduce a function defined for a general complex parameter s,

$$\Gamma(s) \equiv \sum_j \frac{-iZ_j}{s-m_j^2} + \int_{4m_1^2}^\infty d\mu^2\, \sigma(\mu^2) \frac{-i}{s-\mu^2}. \tag{12}$$

Then $G_F(p^2) = \Gamma(s = -p^2 + i\epsilon)$, and so, we obtain the Feynman function by approaching the real $s-$axis from above. We observe two features of $\Gamma(s)$:

1. $\Gamma(s)$ has poles at single-particle mass-squared values: $s = m_j^2$.
2. There is a branch cut beginning at $4m_1^2$ with a discontinuity $\Gamma(r + i\epsilon) - \Gamma(r - i\epsilon) = 2\pi\sigma(r)$.

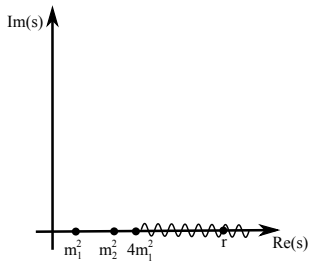

Figure 2: The function $\Gamma(s)$ plotted on the complex plane.

Now, consider

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 + \mathscr{L}_I(\phi, \lambda), \tag{13}$$

and suppose that as $\lambda \longrightarrow 0$, $\mathscr{L} \longrightarrow 0$. At $\lambda = 0$, $G_F^{(0)}(p) = \frac{-i}{p^2+m_0^2-i\epsilon}$. If ϕ does not have any bound states, then, for $\lambda \neq 0$, we have

$$G_F(p) = \frac{-iZ}{p^2 + m^2 - i\epsilon} + \int_{4m^2}^{\infty} d\mu^2\, \sigma(\mu^2)\frac{i}{p^2 + \mu^2 - i\epsilon}. \tag{14}$$

In general, we note the following points:

1. $m^2 \neq m_0^2$: we have a mass renormalization. m_0^2, the bare mass which appears in the Lagrangian, is not necessarily m^2, the physical mass which appears in the propagator.
2. $Z \neq 1$: we have a field-strength renormalization. $Z = |\langle 0|\, \phi(0)\, |k\rangle|^2$. These first two points are generic to interactions, and have nothing to do with ultraviolet divergences.
3. $\phi(0)\,|0\rangle$ also generates multiple-particle states.
4. We define $\phi_{phys} \equiv \sqrt{Z}\phi$, meaning $|\langle 0|\, \phi_{phys}(0)\, |k\rangle|^2 = 1$.

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MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 8

In the last lecture, we showed, for a general interacting theory:

$$G_F(p) = \frac{-iZ}{p^2 + m^2 - i\epsilon} + \int_{4m^2}^{\infty} d\mu^2 \, \sigma(\mu^2) \frac{i}{p^2 + \mu^2 - i\epsilon}, \tag{1}$$

where the first term is the contribution from single-particle states, and the second term is the contribution from multi-particle states. From this, we have that

$$\mathfrak{Im}(iG_F(p)) = \sum_i \pi\delta(p^2 + m_i^2) Z_i + \pi\sigma(-p^2). \tag{2}$$

This is the spectral function, picking out the physical on-shell states. There is one more sum rule we wish to observe. Begin with the canonical quantization:

$$\left[\dot{\phi}(t,\vec{x}), \phi(t,\vec{y})\right] = -i\delta(\vec{x} - \vec{y}). \tag{3}$$

As this operator is just a complex number, we can equate it with its expectation value:

$$\begin{aligned}
\langle 0| \left[\dot{\phi}(t,\vec{x}), \phi(t,\vec{y})\right] |0\rangle =& -i\delta(\vec{x} - \vec{y}) \\
=& \partial_t \langle 0| \phi(t,\vec{x})\phi(t',\vec{y}) |0\rangle|_{t'\to t} - \partial_t \langle 0| \phi(t',\vec{y})\phi(t,\vec{x}) |0\rangle|_{t'\to t} \\
=& \partial_t G_+(x-y)|_{t'\to t} - \partial_t G_+(y-x)|_{t'\to t} \\
=& \int_0^{\infty} d\mu^2 \, \rho(\mu^2) \left[\partial_t G_+^{(0)}(x-y;\, \mu^2)\Big|_{t'\to t} - \partial_t G_+^{(0)}(y-x;\, \mu^2)\Big|_{t'\to t}\right] \\
=& 2\int_0^{\infty} d\mu^2 \, \rho(\mu^2) \, \partial_t G_+^{(0)}(x-y;\, \mu^2)\Big|_{t'\to t}.
\end{aligned}$$

Recall, in the free theory, we have that

$$\partial_t \, G_+^{(0)}(x-y)\Big|_{t'\to t} = -\frac{i}{2}\delta(\vec{x} - \vec{y}), \tag{4}$$

and so, we have that

$$1 = \int_0^{\infty} d\mu^2 \, \rho(\mu^2). \tag{5}$$

Because $\rho(\mu^2) = \sigma(\mu^2) + \sum_i Z_i \delta(\mu^2 - m_i^2)$, where both terms are greater than or equal to zero for all values of μ^2, we have that

$$\int_{4m^2}^{\infty} d\mu^2 \, \sigma(\mu^2) < 1, \;\; Z_i < 1. \tag{6}$$

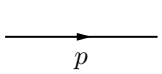
In reality, the above argument may not always hold due to possible ultraviolet divergences. The same discussion can be generalized to a spinor, as seen in the problem set, and to a vector, as we will discuss later.

2.2: AN EXPLICIT EXAMPLE

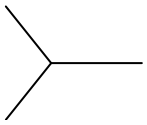
Consider a scalar Lagrangian with a ϕ^3 interaction:

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 + \frac{1}{6}g\phi^3. \tag{7}$$

This leads to the Feynman rules:



$$= \frac{-i}{p^2 + m^2 - i\epsilon},$$



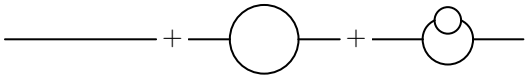
$$= ig, \tag{8}$$

and by dimensional analysis, we find

$$\begin{aligned}[\phi] =& \frac{d-2}{2},\\ [x] =& -1,\\ [g] =& 3 - \frac{d}{2},\end{aligned}$$

where we work in a general space-time dimension d. We note that g is dimensionless in $d = 6$.

$G_F(p)$ =



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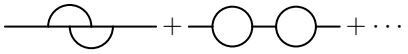
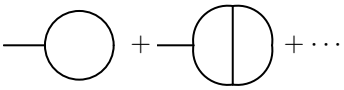


$+ \cdots$ (9)

Note that we do not consider diagrams involving tadpoles:

$\langle 0|\,\phi\,|0\rangle \equiv \phi_0 =$

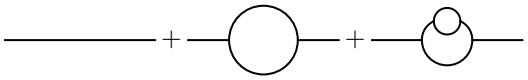


$+ \cdots$ (10)

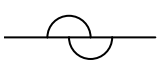
as we can always shift the field definition to $\phi = \phi_0 + \tilde{\phi}$, so that $\langle 0|\,\tilde{\phi}\,|0\rangle = 0$, and the sum of the tadpole sub-diagrams gives zero. Now, we define 1PI, or one-particle irreducible diagrams, as the diagrams which cannot be separated into two disconnected parts by cutting one propagator. We denote the sum of 1PI diagrams by

$i\Pi(p)$ =

=



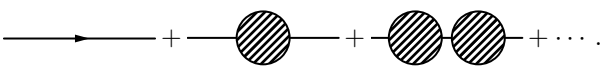
+



$+ \cdots,$ (11)

and so, we have that

$G_F(p) =$



$+ \cdots.$ (12)

That is,

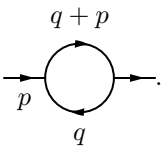
$$\begin{aligned} G_F(p) =& G_F^{(0)} + G_F^{(0)} i\Pi G_F^{(0)} + G_F^{(0)} i\Pi G_F^{(0)} i\Pi G_F^{(0)} + \cdots \\ =& G_F^{(0)} \frac{1}{1 - i\Pi G_F^{(0)}} \\ =& \frac{1}{\left(G_F^{(0)}\right)^{-1} - i\Pi} \\ =& \frac{-i}{p^2 + m_0^2 - \Pi(p) - i\epsilon} \end{aligned}$$

where $\Pi(p)$ is the self-energy. We note that Π is, in fact, a function of p^2 only, by Lorentz symmetry. Before evaluating $\Pi(p^2)$ explicitly to lowest order, we make two remarks:

1. The physical mass, the pole of $G_F(p)$, is given by $p^2 + m_0^2 - \Pi(p^2) = 0$. That is, the physical mass m^2 satisfies
$$m^2 - m_0^2 + \Pi(-m^2) = 0 \tag{13}$$

2. The field renormalization Z, the residue of the pole, is given by expanding around the pole to lowest order:
$$\begin{aligned} iG_F(p)|_{p^2 \approx -m^2} =& \frac{1}{p^2 + m^2 - \Pi'(-m^2)(p^2 + m^2) - i\epsilon} \\ =& \frac{1}{p^2 + m^2 - i\epsilon} \frac{1}{1 - \frac{d\Pi}{dp^2}\Big|_{p^2=-m^2}}, \end{aligned}$$
and so $Z^{-1} = 1 - \frac{d\Pi}{dp^2}\Big|_{p^2=-m^2}$.

We now proceed to evaluate $\Pi(p^2)$ to the lowest order in g:

$i\Pi(p^2) =$ 

. (14)

Using our Feynman rules, we have

$$\begin{aligned} i\Pi(p^2) =& \frac{1}{2}(ig)^2 \int \frac{d^d q}{(2\pi)^d} G_F^{(0)}(q) G_F^{(0)}(q+p) \\ =& \frac{g^2}{2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + m_0^2 - i\epsilon)((q+p)^2 + m_0^2 - i\epsilon)}. \end{aligned}$$

We now evaluate the integral explicitly, using a series of tricks. Firstly, we note the identity, due to Feynman:

$$\frac{1}{a_1 \cdots a_n} = \int_0^1 dx_1 \cdots dx_n \, \delta(x_1 + \cdots + x_n - 1)(n-1)!(x_1 a_1 + \cdots + x_n a_n)^{-n}. \tag{15}$$

Thus, our integrand can be rewritten as

$$\begin{aligned} \frac{1}{(q^2 + m_0^2 - i\epsilon)((q+p)^2 + m_0^2 - i\epsilon)} &= \int_0^1 dx \, \frac{1}{\left[x((q+p)^2 + m_0^2) + (1-x)(q^2 + m_0^2)\right]^2} \\ &= \int_0^1 dx \frac{1}{\left[(q + xp)^2 + D\right]^2} \end{aligned}$$

with $D = m_0^2 + x(1-x)p^2$. Next, we perform a Wick rotation. In Lorentzian signature, the integral is not convenient to evaluate, because of the poles near the integration paths. Thus, we rotate the q_0 contour to the imaginary axis along the direction shown in figure 1. So, we let $q_0 = iq_d$, and therefore $q^2 = q_1^2 + \ldots + q_d^2 = q_E^2$, and $\int d^d q = i \int d^d q_E$.

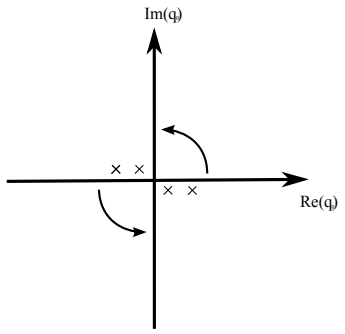


Figure 1: Illustration of the Wick rotation of the variable q_0.

Combining our two results, we have the self-energy in the form

$$i\Pi(p^2) = \frac{ig^2}{2}\int_0^1 dx \int \frac{d^d q_E}{(2\pi)^d}\frac{1}{(q_E^2+D)^2}. \tag{16}$$

We now proceed to the final evaluation. We observe that (16) is convergent only for $d < 4$. We may evaluate it for $d < 4$, then analytically continue its value for $d \geq 4$, treating d as a complex variable. We make use of the general formula

$$\int \frac{d^d q_E}{(2\pi)^d}\frac{(q_E)^a}{(q_E+D)^b} = \frac{\Gamma(b-a-\frac{d}{2})\Gamma(a+\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(b)\Gamma(\frac{d}{2})}D^{-(b-a-\frac{d}{2})}. \tag{17}$$

In this case, we have $b = 2$, $a = 0$. And so,

$$\Pi(p^2) = \frac{g^2}{2}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\int_0^1 dx\,\frac{1}{(m_0^2+x(1-x)p^2)^{2-\frac{d}{2}}}. \tag{18}$$

Example 1: d=3

$$\Pi(p^2) = \frac{g^2}{16\pi}\int_0^1 dx\,\frac{1}{(m_0^2+x(1-x)p^2)^{\frac{1}{2}}}. \tag{19}$$

First, we consider the physical mass to $O(g^2)$. We wish to find the solution to (13). Note that the self-energy is of order g^2, and so the solution is given to $O(g^2)$ by

$$\begin{aligned} m^2 =& m_0^2 - \Pi(-m_0^2) \\ =& m_0^2 - \frac{g^2}{16\pi}\frac{1}{m_0}\int_0^1 dx\,\frac{1}{(1-x(1-x))^{\frac{1}{2}}} \\ =& m_0^2 - \frac{g^2}{16\pi m_0}\log 3. \end{aligned}$$

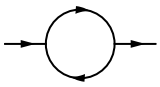
The second term is the mass renormalization due to the interaction, to lowest order. Now, we consider the field renormalization, $Z^{-1} = 1 - \frac{d\Pi}{dp^2}\Big|_{p^2=-m^2}$, and note that again, as $\Pi(p^2)$ is of order g^2, to lowest order we have

$$\begin{aligned} Z^{-1} =& 1 - \frac{d\Pi}{dp^2}\Big|_{p^2=-m_0^2} \\ =& 1 - \frac{g^2}{16\pi}(-\frac{1}{2})\frac{1}{m_0^3}\int_0^1 dx\,\frac{x(1-x)}{(1-x(1-x))^{\frac{3}{2}}} \\ =& \frac{1}{1+\frac{0.23g^2}{32\pi m_0^3}} < 1, \end{aligned}$$

where the integral in dx has been evaluated explicitly. Finally, we note that for $-p^2 \geq 4m_0^2$, we have that $m_0^2 + x(1-x)p^2$ is smaller than zero for a range of x between 0 and 1. Therefore, $\Pi(p^2)$ becomes complex. It is convenient to consider

$$\Pi(s) = \frac{g^2}{16\pi} \int_0^1 dx \, \frac{1}{(m_0^2 - x(1-x)s)^{\frac{1}{2}}} \tag{20}$$

as a function of a complex variable s, with $\Pi(p^2) = \Pi(s = -p^2 + i\epsilon)$. $\Pi(s)$ has a branch point at $s = 4m_0^2$. This is precisely the multiple-particle cut predicted in the general formalism last lecture. Note that $m^2 = m_0^2 + O(g^2)$. We can now understand the physical interpretation of this result:

$i\Pi(p^2) =$  (21)

to lowest order, and when $-p^2 > 4m_0^2$, both the intermediate particles can simultaneously go on-shell. $\mathfrak{Im}(iG_F) = \pi\sigma(-p^2)$ becomes non-zero and $\mathfrak{Im}(-p^2)$ is just the Feynman diagram evaluated with both the intermediate particles on shell, giving a factor of $\delta(p^2 + m^2)$ for each propagator.

8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 9

2.3: REMOVING ULTRAVIOLET DIVERGENCES

Let us consider the same ϕ^3 example, but now in $d = 6$, where the coupling constant g becomes dimensionless.

$$\Pi(p^2) = \frac{g^2}{2}\int_0^1 dx \int \frac{d^d q_E}{(2\pi)^d}\frac{1}{(q_E^2 + D)^2}, \tag{1}$$

with $D = m_0^2 + x(1-x)p^2$, which is quadratically divergent. The integral over the momentum can be expressed in terms of gamma functions, giving

$$\Pi(p^2) = \frac{g^2}{2}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\int_0^1 dx \frac{1}{D^{2-\frac{d}{2}}}. \tag{2}$$

As $d \longrightarrow 6$, this quantity diverges. A similar result holds for quantum electrodynamics. Divergences in physics are, actually, a good thing: they signal missing information in a model, and lead to new physics. In this case, the divergences are emerging at $q_E \longrightarrow \infty$. Thus, there is potentially new short-distance physics, which is to be expected, as in a grand unified theory or a quantum theory of gravity. We need to decide if this new physics is important for what we want to understand and calculate. There are two possibilities: either the new physics is required to make calculations, or it is possible to extract physical results in our regime of interest which are insensitive to the ultraviolet phenomena. It turns out that this second possibility is true, and the procedure is known as renormalization.

In the late 1930s, Dirac, Pauli and Jordan developed quantum electrodynamics. Divergences in the loop diagrams were soon found. Sensitive new experiments, such as the anomalous magnetic moment, $g \neq 2$, and the Lamb shift, meant that these loop corrections were important. A younger generation of physicists, including Feynman, Schwinger, Tomonoga and Dyson, developed the systematics of renormalization, and used it to calculate finite, sensible answers which gave extremely precise and accurate predictions. The understanding of renormalization didn't come until the work of Wilson in the late 60s and early 70s.

In our situation, the divergence in $\Pi(p^2)$ means that, from $m^2 - m_0^2 = -\Pi(m^2)$, the corrections to m_0^2 are infinite, and $Z^{-1} = 1 - \frac{d\Pi}{dp^2}\Big|_{p^2=-m^2}$ is also infinite, so there is an infinite field renormalization. As a clue to how we resolve this, we note that we measure the physical m^2, which is finite. Similarly, if we introduce $\phi_{phys} = \frac{\phi}{\sqrt{Z}}$, then, for ϕ_{phys},

$$G_F(p)|_{p^2\approx -m^2} = \frac{-i}{p^2 + m^2 - i\epsilon}, \tag{3}$$

with $Z_{phys} = 1$, that is, $|\langle 0|\,\phi_{phys}\,|p\rangle|^2 = 1$. ϕ_{phys} is, again, something we directly construct from physical measurements. So, the bare quantities, m_0^2, ϕ are ill-behaved, but this is fine, as they are not directly observable. The strategy, therefore, is to express the Lagrangian in terms of the physical mass and field:

$$\begin{aligned}\mathscr{L} &= -\frac{1}{2}(\partial\phi_B)^2 - \frac{1}{2}m_0^2\phi_B^2 - \frac{1}{6}g_0\phi_B^2 \\ &= -\frac{1}{2}Z_\phi(\partial\phi)^2 - \frac{1}{2}Z_m m^2\phi^2 - \frac{1}{6}Z_g g\phi^2\end{aligned}$$

where ϕ is the physical field, m is the physical mass, and g is the physical coupling. By definition, we have that

$$\langle 0|\,\phi(0)\,|0\rangle = 0,\ |\langle 0|\,\phi(0)\,|p\rangle|^2 = 1. \tag{4}$$

ϕ_B, g_B, and m_B are the bare quantities appearing in the original Lagrangian. They are related to the physical parameters by

$$\phi_B = \sqrt{Z_\phi}\phi,\ m_0^2 = \frac{Z_m}{Z_\phi}m^2,\ g_0 = \frac{Z_g}{Z_\phi^{\frac{3}{2}}}g \tag{5}$$

where Z_ϕ, Z_m and Z_g are potentially divergent. We now write

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_I + \mathscr{L}_{ct} \tag{6}$$

with

$$\begin{aligned}
\mathscr{L}_0 &= -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2, \\
\mathscr{L}_I &= -\frac{1}{6}g\phi^2, \\
\mathscr{L}_{ct} &= -\frac{1}{2}A(\partial\phi)^2 - \frac{1}{2}Bm^2\phi^2 - \frac{1}{6}Cg\phi^2,
\end{aligned}$$

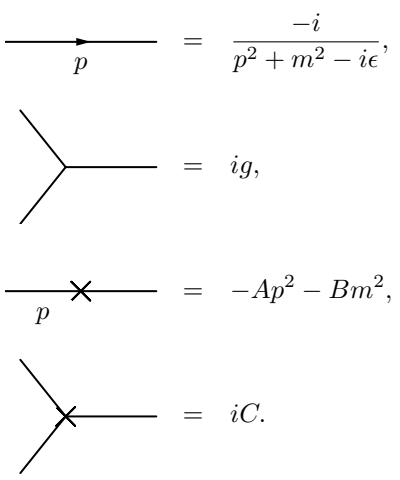
where $A = Z_\phi - 1$, $B = Z_m - 1$, $C = Z_g - 1$.

Remarks

1. A, B and C are potentially divergent, but that is allowed, as they are not observable.

2. Since tree-level diagrams do not change m^2, ϕ or g, the differences between bare and physical quantities come from loop corrections. In perturbation theory, $A, B, C \sim O(g^2)$ and higher.

3. We have a set of modified Feynman rules



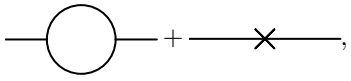
(7)

with the counter-terms treated as additional vertices.

4. A, B, and C will be determined order by order in perturbation theory as follows: again,

$$iG_F(p^2) = \frac{1}{p^2 + m^2 - \Pi(p^2) - i\epsilon}, \tag{8}$$

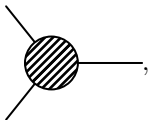
but now, to $O(g^2)$

$\Pi(p^2) =$  (9)

and we have a contribution from the counter-term vertex. We now require

$$\begin{aligned}
\Pi(p^2 = -m^2) &= 0 : \text{the physical mass condition, and} \\
\frac{d\Pi}{dp^2}(p^2 = -m^2) &= 0 : \text{the physical field condition.}
\end{aligned}$$

giving two constraints which can be solved to find A and B. A similar condition for the interaction vertex:

$ig =$  , (10)

can be used to determine C, but we will not discuss that here.

Let us now determine A and B to $O(g^2)$. To deal with divergent quantities, we first need to regularize them, for example by introducing a momentum cutoff :

$$\Pi(p^2) = \frac{g^2}{2}\int_0^1 dx \int^\Lambda \frac{d^d q_E}{(2\pi)^d}\frac{1}{(q_E^2+D)^2} - Ap^2 - Bm^2, \tag{11}$$

taking the limit $\Lambda \longrightarrow \infty$ at the end. Now, $D = m^2 + x(1-x)p^2$; that is, m_0 has been replaced with m. It is often more convenient to use a method known as dimensional regularization. For general d,

$$\Pi(p^2) = \frac{g^2}{2}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\int_0^1 dx \,\frac{1}{D^{2-\frac{d}{2}}} - Ap^2 - Bm^2, \tag{12}$$

and for $d = 6$, $\Gamma(2-\frac{d}{2}) = \Gamma(-1)$ is divergent. We will instead take $d = 6-\epsilon$, taking the limit $\epsilon \longrightarrow 0$ at the end. Note that $[g] = \frac{\epsilon}{2}$ in $d = 6-\epsilon$ dimensions. It is convenient to write $g \longrightarrow g\mu^{\frac{\epsilon}{2}}$, where g is now dimensionless and μ is some arbitrary mass scale. Hence,

$$\Pi(p^2) = \frac{g^2\mu^\epsilon}{2}\frac{\Gamma(-1+\frac{\epsilon}{2})}{(4\pi)^{3-\frac{\epsilon}{2}}}\int_0^1 dx\, D^{1-\frac{\epsilon}{2}} - Ap^2 - Bm^2. \tag{13}$$

The steps are now to expand in ϵ, absorbing the divergent parts into A and B, and then to use the physical mass and field conditions to determine the finite parts of A and B. Introducing $\alpha \equiv \frac{g^2}{(4\pi)^3}$, and using the expansion $\Gamma(1+\frac{\epsilon}{2}) = -\frac{2}{\epsilon} + (\gamma-1) + O(\epsilon)$, where γ is the Euler number, we have

$$\Pi(p^2) = \frac{\alpha}{2}(-\frac{2}{\epsilon})\int_0^1 dx\, D + \frac{\alpha}{2}\left[(\gamma-1)\int_0^1 dx\, D - \int_0^1 dx\, D\log\frac{4\pi\mu^2}{D}\right] - Ap^2 - Bm^2 + O(\epsilon). \tag{14}$$

We note that $\int_0^1 dx\, D = m^2 + \frac{p^2}{6}$. So, the divergence can indeed by cancelled by A and B. Introducing $D_0 = D(p^2 = -m^2) = m^2(1-x(1-x))$, we have

$$\Pi(p^2) = \frac{\alpha}{2}\int_0^1 dx\, D\log(\frac{D}{D_0}) + \frac{\alpha}{2}\left(-\frac{2}{\epsilon} + (\gamma-1) + \log(\frac{D_0}{4\pi\mu^2})\right)(m^2 + \frac{p^2}{6}) - Ap^2 - Bm^2 + O(\epsilon), \tag{15}$$

and so, cancelling the divergent parts into A and B, we have a finite remainder

$$\Pi(p^2) = \frac{\alpha}{2}\int_0^1 dx\, D\log(\frac{D}{D_0}) + ap^2 + bm^2. \tag{16}$$

We now impose the physical mass and physical field conditions, giving

$$\Pi(p^2 = -m^2) = 0 \Rightarrow a = b$$
$$\frac{d\Pi}{dp^2}(p^2 = -m^2) = 0 \Rightarrow a + \frac{\alpha}{2}\int_0^1 dx\, x(1-x) = 0.$$

And so, we find $a = b = -\frac{\alpha}{12}$, and our final result is

$$\Pi(p^2) = \frac{\alpha}{2}\int_0^1 dx\, D\log(\frac{D}{D_0}) - \frac{\alpha}{12}(p^2+m^2) \tag{17}$$

which is completely finite and well-defined.

Remarks:

1. For A and B, we have

$$Ap^2 + Bm^2 = \frac{\alpha}{2}\left(-\frac{2}{\epsilon} + (\gamma - 1) + \log(\frac{D_0}{4\pi\mu^2})\right)(m^2 + \frac{p^2}{6}) + \frac{\alpha}{12}(p^2 + m^2), \tag{18}$$

which is completely determined, and divergent as $\epsilon \longrightarrow 0$. Note that the expression is $O(\alpha)$ where $\alpha = \frac{g^2}{(4\pi)^3}$.

2. While A and B depend on the mass scale μ, which was introduced arbitrarily, physical quantities like $\Pi(p^2)$ do not depend on μ.

3. For large p^2, $D \sim p^2 x(1-x)$, and so

$$\left.\Pi(p^2)\right|_{p^2 \longrightarrow \infty} = \alpha\left[c_1 p^2 + c_2 m^2 + c_3 p^2 \log(\frac{p^2}{m^2})\right]. \tag{19}$$

If $\frac{p^2}{m^2}$ is large, $\alpha \log(\frac{p^2}{m^2})$ can be large and perturbation theory can become invalidated. Later, we will discuss this in the context of the renormalization group.

8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 10

2.4: UNSTABLE PARTICLES AND RESONANCES

We recall from previous lectures that the Feynman propagator for a general interacting theory is given by

$$G_F(p^2) = \Gamma(s = -p^2 + i\epsilon), \tag{1}$$

where

$$\Gamma(s) = \sum_j \frac{iZ_j}{s - m_j^2} + \int_{4m_1^2}^{\infty} d\mu^2\, \sigma(\mu^2) \frac{i}{s - \mu^2}, \tag{2}$$

and the spectral density function is given by

$$\begin{aligned} \rho(\mu^2) =& \frac{1}{\pi} \mathfrak{Im}(iG_F(p^2)\big|_{p^2=-\mu^2}) \\ =& \sum_j Z_j \delta(\mu^2 - m_j^2) + \sigma(\mu^2). \end{aligned}$$

For $\mu^2 > 4m_1^2$, we have that $\rho(\mu^2) = \sigma(\mu^2) = \frac{1}{2\pi}\text{Disc}(\Gamma(s)|_{s=\mu^2})$. The reason for this is that we cannot have

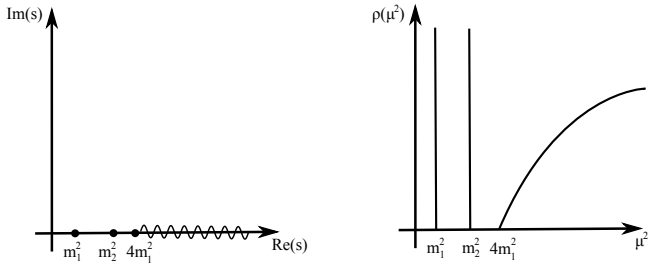


Figure 1: Plots of the function $\Gamma(s)$ on the complex plane and the spectral density function $\rho(\mu^2)$.

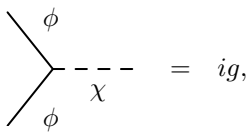
$m_j^2 > 4m_1^2$ for any particle j which interacts with the others. We can imagine changing the parameters of the theory, so that the pole moves beyond the branch point and sits on the branch cut. We will consider what happens in this case. Consider the Lagrangian

$$\mathscr{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}m_\chi^2\chi^2 + \frac{1}{2}g_\phi\phi^3 + \cdots, \tag{3}$$

and examine the Feynman propagator $G_F^\phi(p^2)$. Suppose that $2m_\chi > m_\phi > m_\chi$. Then, perturbatively,

$$iG_F^\phi(p^2) = \frac{1}{p^2 + m_\phi^2 - \Pi^{(\phi)}(p^2) - i\epsilon}. \tag{4}$$

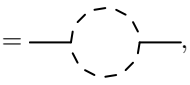
The three-point vertex rule is given by



$$= \; ig, \tag{5}$$

and hence, to lowest order,

$$\Pi^{(\phi)}(p^2) = \tag{6}$$



and we see that $\Pi^{(\phi)}(p^2)$ becomes imaginary if $-p^2 > 4m_\chi^2$. Naively, the pole just sits on the branch cut. But, as we will show, the pole, in fact, moves to the other sheet of the complex plane. Physically, the particle becomes unstable since it can decay $\phi \longrightarrow 2\chi$, where the imaginary part of the pole is proportional to the decay rate. In the Hilbert space of physical states, no states correspond to a single ϕ-particle. It becomes a resonance.

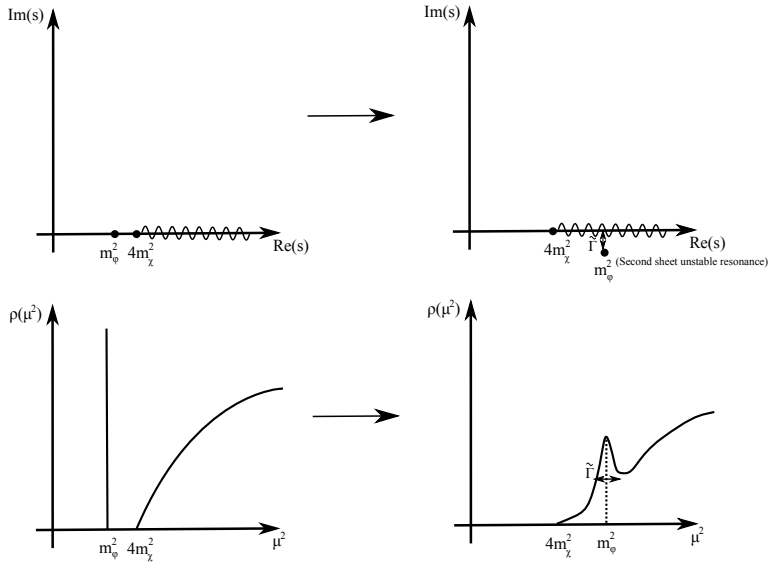


Figure 2: Formation of an unstable resonance for $m_\phi > 2m_\chi$.

From the last lecture, after subtracting divergences using counter-terms,

$$\Pi^{(\phi)}(p^2) = \quad \phi \quad = \frac{\alpha}{2}\int_0^1 dx\, D \log D - a' p^2 - b' m_\phi^2, \tag{7}$$

where $\alpha \equiv \frac{g^2}{(4\pi)^3}$, $D = m_\chi^2 + x(1-x)p^2$, and a' and b' are finite real constants, determined by the physical mass and physical field conditions:

$$\Pi(p^2 = -m^2) = 0,$$
$$\frac{d\Pi}{dp^2}(p^2 = -m^2) = 0.$$

But, for $m_\phi^2 > 4m_\chi^2$, $\log D$ is complex at $p^2 = -m_\phi^2$, and there is no way to satisfy these two conditions for real a' and b'. Therefore, $G_F^{(\phi)}(p^2)$ cannot have a pole on the real axis. Physically, ϕ cannot be a physical single-particle state in $\mathscr{H}$. What can be done to resolve this is to modify the physical mass and physical field conditions to:

$$\mathfrak{Re}\left(\Pi(p^2 = -m^2)\right) = 0,$$
$$\mathfrak{Re}\left(\frac{d\Pi}{dp^2}(p^2 = -m^2)\right) = 0.$$

which gives suitable real solutions for a' and b'. We then have

$$\Pi(p^2) = \frac{\alpha}{2}\int_0^1 dx\, D \log\frac{D}{|D_0|} - \frac{\alpha}{12}(p^2 + m_\phi^2) \tag{8}$$

with $D_0 = D(p^2 = -m_\phi^2) = m_\chi^2 - x(1-x)m_\phi^2$. Let us define $m_\phi\tilde{\Gamma} \equiv \mathfrak{Im}\Pi(p^2)\big|_{p^2=-m_\phi^2}$. Then, near $p^2 = -m_\phi^2$, we have

$$\begin{aligned} iG_F(p^2) =& \frac{1}{p^2 + m_\phi^2 - \Pi(p^2) - i\epsilon} \\ =& \frac{1}{p^2 + m_\phi^2 - im_\phi\tilde{\Gamma}}. \end{aligned}$$

Thus, there is a pole at $p^2 = -m_\phi^2 + im_\phi\tilde{\Gamma}$. In terms of the complex function $\Gamma(s)$, there is a pole at

$$s = m_\phi^2 - im_\phi\tilde{\Gamma}, \tag{9}$$

and so, around the ϕ-mass, the spectral density function takes the form

$$\rho(\mu^2)\big|_{\mu^2\approx m_\phi^2} = \frac{1}{\pi}\mathfrak{Im}(iG_F(p^2)) = \frac{1}{\pi}\frac{m_\phi\tilde{\Gamma}}{(\mu^2 - m_\phi^2)^2 + m_\phi^2\tilde{\Gamma}^2} + \cdots . \tag{10}$$

Now we proceed to calculate $\tilde{\Gamma}$:

$$\mathfrak{Im}(\,\Pi(p^2)\big|_{p^2=-m_\phi^2}) = \frac{\alpha}{2}\int_0^1 dx\,\mathfrak{Im}(D_0\log\frac{D_0}{|D_0|}), \tag{11}$$

where $D_0 = m_\chi^2 - x(1-x)m_\phi^2 - i\epsilon$. Now, $D_0 < 0$ for the range

$$\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4m_\chi^2}{m_\phi^2}} < x < \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4m_\chi^2}{m_\phi^2}}, \tag{12}$$

and in this region, $\log\frac{D_0}{|D_0|} = -i\pi$. Therefore, letting $\beta = \sqrt{1 - \frac{4m_\chi^2}{m_\phi^2}}$, we have

$$\begin{aligned} \mathfrak{Im}(\Pi(-m_\phi^2)) =& -\frac{\alpha\pi}{2}\int_{\frac{1}{2}(1-\beta)}^{\frac{1}{2}(1+\beta)} dx\,(m_\chi^2 - x(1-x)m_\phi^2) \\ =& \frac{\pi\alpha}{12}m_\phi^2\beta^3, \end{aligned}$$

and hence,

$$\tilde{\Gamma} = \frac{\pi\alpha}{12}m_\phi\beta^3. \tag{13}$$

In the problem set, we will see that when $m_\phi > 2m_\chi$, we have

$$\tilde{\Gamma}_{\phi\to 2\chi} = \frac{\pi\alpha}{12}m_\phi(1 - \frac{4m_\chi^2}{m_\phi^2})^{\frac{3}{2}}, \tag{14}$$

and so $\tilde{\Gamma} = \tilde{\Gamma}_{\phi\to 2\chi}$. We now consider a couple of additional points.

Remarks:

1. Consider the function $\log z$. The function increases by 2π as we wrap around the origin each time. This introduces a discontinuity if we try to plot the function on one sheet of the complex plane. Instead, we have to plot the function on multiple sheets, introducing a branch cut, for example on the first sheet at $\mathfrak{Im}\,(\log(-x+i\epsilon)) = i\pi$, $\mathfrak{Im}\,(\log(-x-i\epsilon)) = -i\pi$. Similarly, with $\Gamma(s)$, there is a branch cut at $x = 4m_\chi^2$. When we calculated

$$G_F(p^2) = \frac{1}{p^2 + m_\phi^2 - im_\phi\tilde{\Gamma}} \tag{15}$$

we treated it as an analytic function across the real axis. In other words, we have done the analytic continuation below the cut, and so, the pole lies on the second sheet.

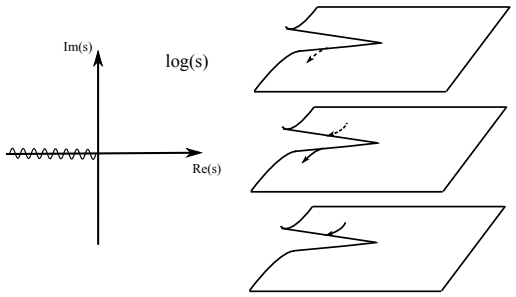


Figure 3: The branch cut in $\log(s)$ and the additional sheets needed to plot the function smoothly.

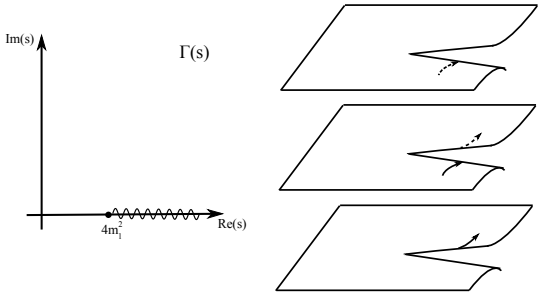
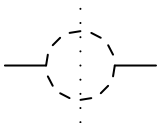
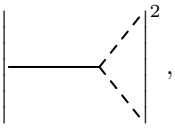


Figure 4: The branch cut in $\Gamma(s)$ with the second sheet on which the pole lies.

2. One can show that, as a consequence of the optical theorem,

$\mathfrak{Im}(\Pi(p^2)) =$ 

$=$ 

, (16)

where the propagators are put on shell, replaced by $2\pi\delta(p_1^2 + m_\chi^2)\Theta(p_1^0)$. This will be discussed in the problem set.

3. Considering the propagation of an unstable particle, for $t > 0$,

$$G_F(t,\vec{p}) = \langle\phi(t,\vec{p})\phi(0,\vec{p})\rangle = \int \frac{dw}{2\pi}\, e^{-i\omega t}\frac{1}{p^2 + m^2 - im\tilde{\Gamma}}, \tag{17}$$

with $p^2 = -\omega^2 + \vec{p}^2$. Hence,

$$G_F(t,\vec{p}) = \frac{e^{-i\sqrt{\omega_{\vec{p}}^2 - im\tilde{\Gamma}}t}}{\sqrt{\omega_{\vec{p}}^2 - im\tilde{\Gamma}}} \tag{18}$$

with $\omega_{\vec{p}} \equiv \sqrt{\vec{p}^2 + m^2}$. Now, for the case that $\tilde{\Gamma} \ll \frac{\omega_{\vec{p}}^2}{m}$, this reduces to

$$\begin{aligned} G_F(t,\vec{p}) =& \frac{1}{2\omega_{\vec{p}}} e^{-i\omega_{\vec{p}}t - \frac{m\tilde{\Gamma}}{2\omega_{\vec{p}}}t} \\ =& \frac{1}{2\omega_{\vec{p}}} e^{-i\omega_{\vec{p}}t - \frac{t}{2\tau(\vec{p})}} \end{aligned}$$

where we have defined the lifetime $\tau(\vec{p}) \equiv \frac{\omega_{\vec{p}}}{m}\tilde{\Gamma}^{-1} = \frac{1}{\sqrt{1-v^2}}\tilde{\Gamma}^{-1}$, which we recognise as the effect of time dilation, where $\tilde{\Gamma}^{-1}$ is the lifetime of the particle in its rest frame.

8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 11

2.5: S-MATRIX ELEMENTS AND LSZ REDUCTION

Having studied in detail the general structure of two-point functions, let us now look at the general structure of higher-point functions. These are of course much more complicated, and the power of Lorentz and translational symmetries becomes much more limited. Nevertheless, there are some important statements to be made. We again consider a scalar field for illustration. Generalizations to spinors, vectors and multiple fields do not contain additional conceptual insights.

We consider

$$G_F(x_1,\ldots,x_n) \equiv \langle 0|\, T(\phi(x_1)\ldots\phi(x_n))\,|0\rangle\,, \tag{1}$$

where $|0\rangle$ is the exact vacuum of the interacting state, and the Fourier transform is given by

$$G_F(p_1,\ldots,p_n) \equiv \int d^4x_1\ldots d^4x_n\, e^{-i(p_1.x_1+\ldots+p_n.x_n)} G_F(x_1,\ldots,x_n), \tag{2}$$

where $p_i^\mu = (\omega_i, \vec{p}_i)$. Now, by translational invariance, we have that $G_F(x_1,\ldots,x_n) = G(0, x_2-x_1,\ldots,x_n-x_1)$, and so

$$G_F(p_1,\ldots,p_n) \propto (2\pi)^4\delta^{(4)}(p_1+\ldots+p_n). \tag{3}$$

Now, if we consider any combination of these momenta, for example

$$p \equiv p_1+p_2+\ldots+p_r = -(p_{r+1}+\ldots+p_n), \quad 1\le r\le n-1, \tag{4}$$

then $G_F(p_1,\ldots,p_n)$ has a pole at $p^2=-m_i^2$, where m_i is the mass of a single-particle state.

Proof: Let us first consider $r=1$, that is, consider

$$G_F(p,y_1,\ldots,y_{n-1}) = \int d^4x\, e^{-ip.x}\,\langle 0|\, T(\phi(x)\phi(y_1)\ldots\phi(y_{n-1}))\,|0\rangle\,. \tag{5}$$

The integration over $x^0=t$ can be separated into three regions:

$$\begin{aligned}\int dt &= \int_{\mathbf{I}} dt + \int_{\mathbf{II}} dt + \int_{\mathbf{III}} dt,\\ &= \int_{T_+}^{\infty} dt + \int_{T_-}^{T_+} dt + \int_{-\infty}^{T_-} dt\end{aligned}$$

where $T_- < y_1^0,\ldots y_{n-1}^0 < T_+$. In region **I**,

$$G_F(x,y_1,\ldots y_{n-1}) = \langle 0|\,\phi(x)T(\phi(y_1)\ldots\phi(y_{n-1}))\,|0\rangle\,. \tag{6}$$

We can now use the same trick as in the case of the two-point function, inserting a complete set of physical states:

$$1 = \sum_n |n\rangle\langle n| = \sum_j \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}^{(j)}}\left|j,\vec{k}\right\rangle\left\langle j,\vec{k}\right| + \text{multi-particle states}, \tag{7}$$

where $\omega_{\vec{k}}^{(j)} = \sqrt{\vec{k}^2+m_j^2}$. For simplicity, let us consider a single species. Then, we have that

$$\begin{aligned}G_F(x,y_1,\ldots,y_{n-1}) &= \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}\left\langle 0\,|\phi(x)|\,\vec{k}\right\rangle\left\langle \vec{k}\,|T(\phi(y_1)\ldots\phi(y_{n-1}))|\,0\right\rangle\\ &= \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}e^{ik.x}\sqrt{Z}\left\langle \vec{k}\,|T(\phi(y_1)\ldots\phi(y_{n-1}))|\,0\right\rangle,\end{aligned}$$

and so, for the integral over region I, we have

$$\begin{aligned}
\mathbf{I} &= \sqrt{Z}\int_{T_+}^{\infty} dt \int d\vec{x} e^{i\omega t - i\vec{p}.\vec{x}} \int \frac{d^3\vec{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{k}}}\left\langle \vec{k}\left|T(\ldots)\right|0\right\rangle \\
&= \sqrt{Z}\frac{1}{2\omega_{\vec{p}}}\frac{ie^{i(\omega-\omega_{\vec{p}})T_+}}{\omega-\omega_{\vec{p}}+i\epsilon}\left\langle\vec{p}\right|T(\ldots)\left|0\right\rangle \\
&\xrightarrow[\omega\to\omega_{\vec{p}}]{} \sqrt{Z}\frac{-i}{p^2+m^2-i\epsilon}\left\langle\vec{p}\right|T(\ldots)\left|0\right\rangle .
\end{aligned}$$

Similarly, for the integral over region **III**, we find

$$\mathbf{III} \xrightarrow[\omega\to-\omega_{\vec{p}}]{} \sqrt{Z}\frac{-i}{p^2+m^2-i\epsilon}\left\langle 0\right|T(\ldots)\left|-\vec{p}\right\rangle . \tag{8}$$

Region **II** is a compact integral, so it does not have singular behaviour. So we conclude that, as a function of p,

$$\begin{aligned}
G_F(p,\ldots) &\xrightarrow[\omega\to\omega_{\vec{p}}]{} \frac{-i\sqrt{Z}}{p^2+m^2-i\epsilon}\left\langle\vec{p}\right|T(\ldots)\left|0\right\rangle , \\
&\xrightarrow[\omega\to-\omega_{\vec{p}}]{} \frac{-i\sqrt{Z}}{p^2+m^2-i\epsilon}\left\langle 0\right|T(\ldots)\left|-\vec{p}\right\rangle .
\end{aligned}$$

The above argument can be generalized to any r. Consider $p = p_1 + \ldots + p_r = -(p_{r+1} + \ldots + p_n)$. Among all possible orderings of $t_1, \ldots, t_n$, we consider those with $\min\{t_1, \ldots, t_r\} > \max\{t_{r+1}, \ldots, t_n\}$. In this case, we have

$$G_F(x_1,\ldots,x_n) = \Theta(\tau)\left\langle 0\right|T(\phi(x_1)\ldots\phi(x_r))T(\phi(x_{r+1})\ldots\phi(x_n))\left|0\right\rangle + \text{other orderings}, \tag{9}$$

with $\tau = \min\{t_1,\ldots,t_r\} - \max\{t_{r+1},\ldots,t_n\}$. Hence, inserting a complete set of intermediate states, including one-particle states, we have

$$G_F(x_1,\ldots,x_n) = \Theta(\tau)\int\frac{d^3\vec{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{k}}}\left\langle 0\left|T(\phi(x_1)\ldots\phi(x_r)\right|\vec{k}\right\rangle\left\langle\vec{k}\left|T(\phi(x_{r+1})\ldots\phi(x_n))\right|0\right\rangle , \tag{10}$$

and we proceed to take the Fourier transform $G_F(x_1,\ldots,x_n) \longrightarrow G_F(p_1,\ldots,p_n)$. The analysis of the Fourier transform is a bit more intricate than the $r = 1$ case. The details are left to the reader; they can be found in *Weinberg, Volume I*, §10.2. The result is that

$$G_F(p_1,\ldots,p_n) \xrightarrow[p^0\to E_{\vec{p}}]{} (2\pi)^4\,\delta^{(4)}(p_1+\ldots+p_n)\,\frac{-i}{p^2+m^2-i\epsilon}M_{0|\vec{p}}(p_2,\ldots,p_r)M_{\vec{p}|0}(p_{r+2},\ldots,p_n) \tag{11}$$

where

$$M_{0|\vec{p}}(p_2,\ldots,p_r) = \int d^4y_2\ldots d^4y_r\, e^{-ip_2\cdot y_2-\ldots-ip_r\cdot y_r}\left\langle 0\right|T(\phi(0)\phi(y_2)\ldots\phi(y_r))\left|\vec{p}\right\rangle , \tag{12}$$

and

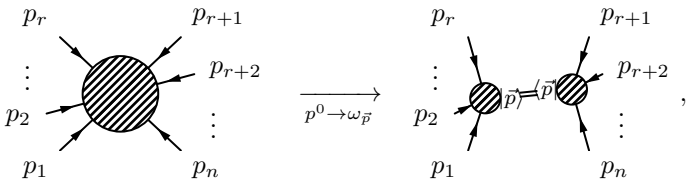
$$M_{\vec{p}|0}(p_{r+2},\ldots,p_n) = \int d^4y_{r+2}\ldots d^4y_n\, e^{-ip_{r+2}\cdot y_{r+2}-\ldots-ip_n\cdot y_n}\left\langle\vec{p}\right|T(\phi(0)\phi(y_{r+2})\ldots\phi(y_n))\left|0\right\rangle . \tag{13}$$

□

Remarks:

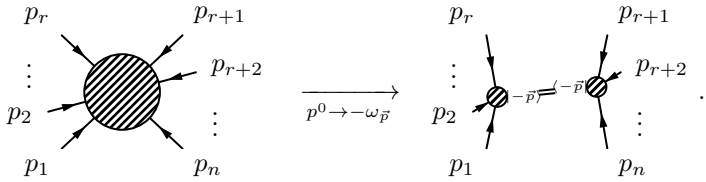
1. The result is generally valid for any interacting theory, and is non-perturbative in nature. In particular, as in the case of the two-point function, the single-particle states do not have to correspond to fields which appear in the Lagrangian. Further, ϕ does not have to be a fundamental field appearing in the Lagrangian: the same conclusion applies if one uses composite operators. For example, in quantum chromodynamics, pions can appear as such a pole.

2. The result is physically intuitive. Diagramatically, it can be expressed as, when $p = p_1 + \ldots + p_r$ is close to on-shell for a particle of mass m,



(14)

where the internal propagator contributes a factor of $\frac{-i}{p^2+m^2-i\epsilon}$. Similarly,



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2.5.2: LSZ Reduction

Now, let us take p_1, p_2 to be simultaneously on-shell. That is, $p_1^0 \approx -\omega_{\vec{p}_1}$, $p_2^0 \approx -\omega_{\vec{p}_2}$. Then the Green's function is given by

$$G_F(p_1, p_2, \ldots, p_n) \longrightarrow \frac{-i\sqrt{Z_1}}{p^2 + m_1^2 - i\epsilon} \frac{-i\sqrt{Z_2}}{p^2 + m_2^2 - i\epsilon} \langle \vec{p} | \, T(\phi_3(x_3) \ldots \phi_n(x_n)) \, | -\vec{p}_1, -\vec{p}_2 \rangle \, . \tag{16}$$

If we now take $p_n^0 \approx \omega_{\vec{p}_3}, \ldots, p_n^0 \approx \omega_{\vec{p}_n}$, we find

$$G_F(p_1, p_2, \ldots, p_n) \longrightarrow \prod_{j=1}^{n} \frac{-i\sqrt{Z_j}}{p^2 + m_j^2 - i\epsilon} \, \langle \vec{p}_3, \ldots, \vec{p}_n | \, -\vec{p}_1, -\vec{p}_2 \rangle \, , \tag{17}$$

giving the $S-$matrix element $\langle \vec{p}_3, \ldots, \vec{p}_n | \, -\vec{p}_1, -\vec{p}_2 \rangle$. This suggests that we can calculate the S-matrix elements using Feynman diagrams as follows:

1. Consider all Feynman diagrams for the Green's function $G(p_1, \ldots, p_n)$.

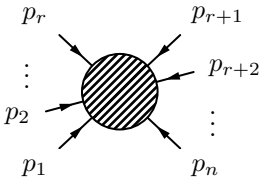
2. Put all the external momenta on shell:

$$\begin{cases} p_i^0 \to -\omega_{\vec{p}_i} & \text{for initial momenta,} \\ p_f^0 \to \omega_{\vec{p}_f} & \text{for final momenta.} \end{cases} \tag{18}$$

3. Obtain amptutated amplitudes by discarding the external propagators.

4. Since an external propagator behaves as $\frac{-i\sqrt{Z}}{p^2+m^2-i\epsilon}$ near the mass-shell, we have

$$\langle f | \, i \rangle = \prod_{j=1}^{n} \sqrt{Z_j} \tag{19}$$



where the shaded vertex denotes all amputated diagrams.

2.6: THE OPTICAL THEOREM

The optical theorem is a simple consequence of the unitarity of the S-matrix. Since

$$S^\dagger S = 1, \tag{20}$$

where it is convential to write $S = 1 + iT$, we have that

$$-i(T - T^\dagger) = T^\dagger T. \tag{21}$$

We now take the matrix elements of this equation between some states

$$-i \, \langle b | \, (T - T^\dagger) \, | a \rangle = \langle b | \, T^\dagger T \, | a \rangle \tag{22}$$

where we have $\langle b|\, T\, |a\rangle \equiv M(a \to b)$, and so $\langle b|\, T^\dagger\, |a\rangle = M(b \to a)*$. Inserting a complete set of states into the right-hand side of this equation, we obtain

$$\begin{aligned} \langle b|\, T^\dagger T\, |a\rangle &= \sum_n \langle b|\, T^\dagger\, |n\rangle \langle n|\, T\, |a\rangle \\ &= \sum_n M(a \to n) M(b \to n)*, \end{aligned}$$

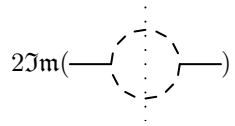
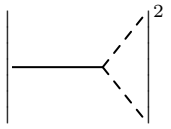
and so we obtain the result

$$-i\left[M(a \to b) - M(b \to a)*\right] = \sum_n |M(a \to n)|^2 . \tag{23}$$

In particular, if we take $a = b$, we find

$$2\mathfrak{Im}(M(a \to a)) = \sum_n |M(a \to n)|^2 , \tag{24}$$

relating the imaginary part of forward scattering to the total cross section. This result can also be applied order-by-order in perturbation theory. Recall the example we discussed in lecture 10:

$2\mathfrak{Im}($  $) \quad = \quad$  . (25)

8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 12

3: GENERAL ASPECTS OF QUANTUM ELECTRODYNAMICS

3.1: RENORMALIZED LAGRANGIAN

Consider the Lagrangian of quantum electrodynamics in terms of the bare quantities:

$$\mathscr{L} = -\frac{1}{4}F^B_{\mu\nu}F^{\mu\nu}_B - i\bar{\psi}_B(\gamma^\mu(\partial_\mu - ie_B A^B_\mu) - m_B)\psi_B. \tag{1}$$

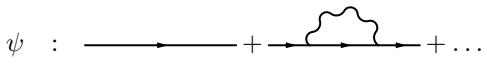
We use the convention:

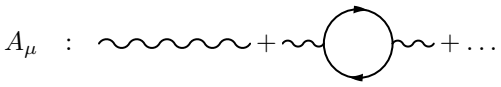
$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \tag{2}$$

$$\gamma_0^2 = -1, \quad \gamma_0^\dagger = -\gamma_0, \quad \gamma_i^\dagger = \gamma_i,$$
$$\bar{\psi} = \psi^\dagger\gamma_0, \quad \not{k} = k^\mu\gamma_\mu, \quad \not{k}^2 = k^2.$$

where, in four dimensions, $(\eta^{\mu\nu}) = \mathrm{diag}(-1, 1, 1, 1)$. We expect to find the mass and field renormalizations.

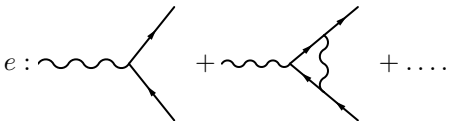
Note: we will omit the "B" signifying bare quantities in what follows.





(3)

along with vertex corrections



(4)

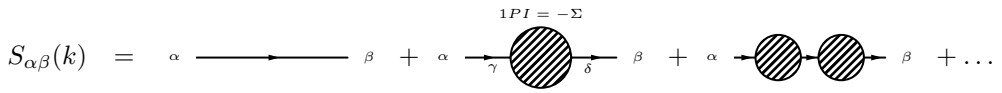
We will look at how to introduce renormalized quantities.

3.1.1: Fermion self-energy

We have that

$$S_{\alpha\beta}(x) = \langle 0|\, T(\psi_\alpha(x)\bar{\psi}_\beta(0)\,|0\rangle\,, \tag{5}$$

and

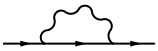


$$= S_0(k) + S_0(-\Sigma)S_0 + S_0(-\Sigma)S_0(-\Sigma)S_0 + \ldots$$
$$= S_0\frac{1}{1+\Sigma S_0}, \tag{6}$$

where we have omitted the spinor indices in the second and third lines: these are to be read as matrix equations. Hence, we have that

$$S^{-1} = S_0^{-1} + \Sigma. \tag{7}$$

Recall that

$$S_0 = \frac{-1}{i\not{k} + m_B}, \quad -\Sigma = \text{  } + \ldots, \tag{8}$$

and so

$$S^{-1} = -(i\not{k} + m_B) + \Sigma, \tag{9}$$

and we have for the fully interacting two-point function

$$S = \frac{-1}{i\not{k} + m_B - i\epsilon - \Sigma}. \tag{10}$$

Note that $\Sigma = \Sigma(\not{k})$, since it can only depend on $\not{k}$ and k^2. Even though it is a function of matrix, we can treat it as an ordinary function. The physical mass is defined by $\not{k} = im$, so that

$$-m + m_B - \Sigma(im) = 0. \tag{11}$$

We note that, again, Σ will be divergent. Near the pole, we have that

$$S \approx \frac{-Z_2}{i\not{k} + m - i\epsilon} \tag{12}$$

with $Z_2^{-1} = 1 + i \left.\frac{d\Sigma}{d\not{k}}\right|_{\not{k}=im}$. The relations between the bare and physical quantities are given by

$$m_B = m + \delta m, \quad \psi_B = \sqrt{Z_2}\psi \tag{13}$$

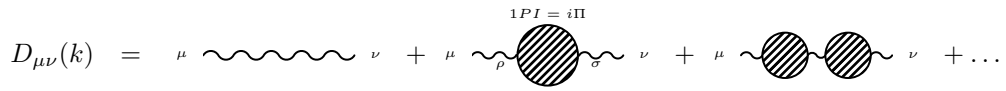
where $\delta m = \Sigma(im)$ and ψ is the renormalized field.

3.1.2: Photon self-energy

Similarly, we have that

$$D_{\mu\nu}(x) = \langle 0 | \, T(A_\mu^B(x) A_\nu^B(0) \, | 0 \rangle \,, \tag{14}$$

and

$$D_{\mu\nu}(k) = \text{  }$$

$$\begin{aligned} &= D_0(k) + D_0(i\Pi)D_0 + D_0(i\Pi)D_0(i\Pi)D_0 + \ldots \\ &= D_0 \frac{1}{1 - i\Pi D_0} \end{aligned} \tag{15}$$

Hence,

$$(iD)^{-1} = (iD_0)^{-1} - \Pi. \tag{16}$$

Recall that

$$\begin{aligned} iD_0^{\mu\nu} &= \frac{1}{k^2 - i\epsilon}\left[\eta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right] \\ &= \frac{1}{k^2 - i\epsilon}\left[P_T^{\mu\nu} + \xi P_L^{\mu\nu}\right], \end{aligned}$$

where we have defined the transverse projector $P_T^{\mu\nu} \equiv \eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$, and the longitudinal projector $P_L^{\mu\nu} \equiv \frac{k^\mu k^\nu}{k^2}$. Note that it is not a coincidence that the propagator can be built from these two tensors, $\eta^{\mu\nu}$ and $k^\mu k^\nu$: they are the only two two-tensors allowed by symmetry. These projectors satisfy the properties

$$P_T^{\mu\nu} P_{\nu\lambda}^T = P_{T\,\nu}^{\mu}, \quad P_L^{\mu\nu} P_{\nu\lambda}^L = P_{L\,\lambda}^{\mu}, \quad P_T^{\mu\nu} P_{\nu\lambda}^L = 0. \tag{17}$$

Note: T and L are just labels here, and the placing of these indices does not carry meaning. Hence, we have that

$$(iD_0)^{-1} = k^2 \left[P_T^{\mu\nu} + \frac{1}{\xi} P_L^{\mu\nu} \right]. \tag{18}$$

We may also expand $\Pi^{\mu\nu}$ as

$$\begin{aligned} \Pi^{\mu\nu} &= P_T^{\mu\nu} f_T(k^2) + P_L^{\mu\nu} f_L(k^2) \\ &= \eta^{\mu\nu} f_T + \frac{k^\mu k^\nu}{k^2}(f_L - f_T). \end{aligned} \tag{19}$$

Therefore,

$$(iD)^{-1} = P_T^{\mu\nu}(k^2 - f_T) + P_L^{\mu\nu}(\frac{k^2}{\xi} - f_L), \tag{20}$$

and we have for the full interacting photon two-point function,

$$D = -i \left[P_T^{\mu\nu} \frac{1}{k^2 - f_T} + P_L^{\mu\nu} \frac{1}{\frac{k^2}{\xi} - f_L} \right]. \tag{21}$$

We observe that if $f_{T,L}(k^2 = 0) \neq 0$, a mass will be generated for the photon. Because $\Pi^{\mu\nu}$ comes from 1PI diagrams, it should not be singular at $k^2 = 0$, and so $f_L - f_T = O(k^2)$, as $k \longrightarrow 0$. We will show that gauge invariance ensures that no mass is generated from the loop corrections.

3.1.3: Ward identities

Consider the path integral for the generating functional:

$$Z\left[J_\mu, \eta, \bar{\eta}\right] = \int \mathfrak{D}A_\mu \mathfrak{D}\psi \mathfrak{D}\bar{\psi} e^{iS\left[A_\mu, \psi, \bar{\psi}\right]} \tag{22}$$

where $S = S_{QED} + \int d^4x \, J_\mu A_B^\mu + \bar{\eta}\psi_B + \bar{\psi}_B \eta$, where we note explicitly these couplings are in terms of bare quantities.

$$\mathscr{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i\bar{\psi}(\gamma^\mu D_\mu - m)\psi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2. \tag{23}$$

We define the generating functional for connected diagrams, $W\left[J_\mu, \eta, \bar{\eta}\right]$, by

$$Z\left[J_\mu, \eta, \bar{\eta}\right] = e^{iW\left[J_\mu, \eta, \bar{\eta}\right]}. \tag{24}$$

For example,

$$\begin{aligned} \langle 0| T(\psi_\alpha(x)\bar{\psi}_\beta(y)) |0\rangle &= i \left. \frac{\delta^2 W\left[J_\mu, \eta, \bar{\eta}\right]}{\delta\eta_\alpha(x)\delta\eta_\beta(y)} \right|_{J=\eta=\bar{\eta}=0}, \\ \langle 0| T(A_\mu^B(x) A_\nu^B(y)) |0\rangle &= i \left. \frac{\delta^2 W\left[J_\mu, \eta, \bar{\eta}\right]}{\delta J^\mu(x)\delta J^\nu(y)} \right|_{J=\eta=\bar{\eta}=0}. \end{aligned}$$

Recall, for infinitesimal gauge transformations, $\delta A_\mu = \partial_\mu \lambda$, $\delta\psi = ie_B\lambda\psi$, and $\delta\bar{\psi} = -ie_B\lambda\bar{\psi}$. Consider a change of variables in the path integral:

$$\begin{aligned} A_\mu &\longrightarrow A'_\mu = A_\mu + \delta A_\mu, \\ \psi &\longrightarrow \psi' = \psi + \delta\psi, \\ \bar{\psi} &\longrightarrow \bar{\psi}' = \bar{\psi} + \delta\bar{\psi}. \end{aligned}$$

Then we have

$$\int \mathfrak{D}A'_\mu \mathfrak{D}\psi' \mathfrak{D}\bar{\psi}' e^{iS\left[A'_\mu, \psi', \bar{\psi}'\right]} = \int \mathfrak{D}A_\mu \mathfrak{D}\psi \mathfrak{D}\bar{\psi} e^{iS\left[A_\mu, \psi, \bar{\psi}\right]}, \tag{25}$$

as this is just of a change of the dummy integration variables. Note that the measure is unchanged by this shift:

$$\mathfrak{D}A'_\mu \mathfrak{D}\psi' \mathfrak{D}\bar{\psi}' = \mathfrak{D}A_\mu \mathfrak{D}\psi \mathfrak{D}\bar{\psi}, \tag{26}$$

and the action for the two sets of variables are related by

$$S\left[A'_\mu,\psi',\bar\psi'\right]=S\left[A'_\mu,\psi',\bar\psi'\right]-\frac{1}{\xi}\int d^4x\,\partial_\mu A^\mu\partial^2\lambda+\int d^4x\,J_\mu\partial^\mu\lambda+ie_B\lambda\bar\eta\psi-ie_B\lambda\bar\psi\eta. \tag{27}$$

Hence, we must have

$$\int d^4x\,\lambda(x)\int\mathfrak{D}A_\mu\mathfrak{D}\psi\mathfrak{D}\bar\psi e^{iS[A,\psi,\bar\psi]}\left[-\frac{1}{\xi}\partial^2\partial_\mu A^\mu-\partial_\mu J^\mu+ie_B(\bar\eta\psi-\bar\psi\eta)\right]=0. \tag{28}$$

Since

$$\begin{aligned}A_\mu(x)&\sim -i\frac{\delta Z}{\delta J^\mu(x)}=Z\frac{\delta W}{\delta J^\mu(x)},\\ \psi(x)&\sim -i\frac{\delta Z}{\delta\bar\eta(x)}=Z\frac{\delta W}{\delta\bar\eta(x)},\\ \bar\psi(x)&\sim -i\frac{\delta Z}{\delta\eta(x)}=Z\frac{\delta W}{\delta\eta(x)},\end{aligned}$$

we have that

$$\frac{1}{\xi}\partial^2\partial^\mu\left.\frac{\delta^2W}{\delta J^\mu(x)\delta J^\nu(y)}\right|_{J=\eta=\bar\eta=0}+\partial_\nu\delta^{(4)}(x-y)=0, \tag{29}$$

that is,

$$\frac{i}{\xi}\partial^2\partial^\mu D_{\mu\nu}(x-y)+\partial_\nu\delta^{(4)}(x-y)=0, \tag{30}$$

or, written in momentum-space,

$$-\frac{i}{\xi}k^2k^\mu D_{\mu\nu}(k)+k_\nu=0. \tag{31}$$

If we now write

$$D_{\mu\nu}(k)=P^T_{\mu\nu}D_T(k^2)+P^L_{\mu\nu}D_L(k^2), \tag{32}$$

with $k^\mu P^L_{\mu\nu}=k_\nu$, the Ward identity reduces to

$$\frac{-i}{\xi}k^2k_\nu D_L(k^2)+k_\nu=0, \tag{33}$$

and so

$$D_L(k^2)=-\frac{i\xi}{k^2}, \tag{34}$$

and the longitudinal part of the two-point function is completely determined. Comparing this with (21), we find that $f_L(k^2)=0$, and we thus conclude that $\Pi^{\mu\nu}$ is purely transverse. That is, from (19), we have that

$$\Pi^{\mu\nu}=\left(\eta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)f_T(k^2). \tag{35}$$

For $\Pi^{\mu\nu}(k)$ to be non-singular at $k=0$, we must have

$$f_T(k^2)=k^2\Pi(k^2), \tag{36}$$

where $\Pi(0)$ is non-singular. Hence, for the two-point function in the interacting theory, we have

$$D_{\mu\nu}=\frac{-i}{k^2-i\epsilon}\left[\frac{P^T_{\mu\nu}}{1-\Pi(k^2)}+\xi P^L_{\mu\nu}\right]. \tag{37}$$

Remarks:

1. The longitudinal part of $D_{\mu\nu}$ does not receive any loop corrections: it is completely determined by the Ward identities. The physics should not depend on this part. For example, in the Landau gauge, $\xi=0$, $D_{\mu\nu}$ is purely transverse.

2. Since $\Pi(k^2)$ is non-singular at $k^2 = 0$, the photon remains massless to all orders. There are exceptions to this: it is not true in quantum electrodynamics in $1+1$ dimensions, or in theories where an additional Higgs field is introduced.

3. The residue at the $k^2 = 0$ pole is given by $Z_3^{-1} = 1 - \Pi(0)$, and we have that

$$iD^T_{\mu\nu} \approx \frac{Z_3}{k^2 - i\epsilon} P^T_{\mu\nu} \tag{38}$$

near $k^2 = 0$. The renormalized field is given by $A^B_\mu = \sqrt{Z_3} A_\mu$.

8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

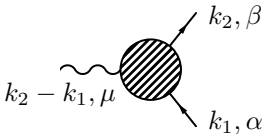
Lecture 13

We continue our analysis of renormalization in quantum electrodynamics from last lecture.

3.1.4: Charge Renormalization

Consider the vertex corrections:

$$\begin{aligned} G_{\mu\alpha\beta}(k_1,k_2) &\equiv \int d^4y_1 d^4y_2\, e^{ik_1.y_1-ik_2.y_2} \langle 0|\, T(A_\mu(0)\psi_\alpha(y_1)\bar\psi_\beta(y_2))\,|0\rangle \\ &= \end{aligned}$$



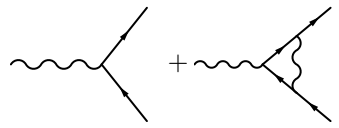
, (1)

where, again, these are defined in terms of bare quantities. We introduce an effective vertex, Γ, defined by

$$G_{\mu\alpha\beta}(k_1,k_2) = D_{\mu\nu}(k_2-k_1)S_{\alpha\delta}(k_1)\Gamma^\nu_{\delta\lambda}(k_1,k_2)S_{\lambda\beta}(k_2). \tag{2}$$

In perturbation theory,

$$\Gamma^\mu_{\alpha\beta}(k_1,k_2) =$$



$$= -ie_B\gamma^\mu_{\alpha\beta} + \dots. \tag{3}$$

We note that only 1PI diagrams contribute, by the definition. We will now show that gauge invariance, in the form of the Ward identities, puts important constraints on the structure of $\Gamma^\mu_{\alpha\beta}$. Acting on the generating functional for connected diagrams, and setting $J_\mu = \eta = \bar\eta = 0$, we have

$$\frac{1}{\xi}\partial^2\partial^\mu \left.\frac{\delta^3 W}{\delta J_\mu(x)\delta\bar\eta_\alpha(y_1)\delta\eta_\beta(y_2)}\right|_{J=\eta=\bar\eta=0} = ie_B\left[\delta^{(4)}(x-y_1)\frac{\delta^2 W}{\delta\bar\eta_\alpha(x)\delta\eta_\beta(y_2)} - \delta^{(4)}(x-y_2)\frac{\delta^2 W}{\delta\bar\eta_\alpha(y_1)\delta\eta_\beta(x)}\right]_{J=\eta=\bar\eta=0}, \tag{4}$$

or, equivalently,

$$\frac{1}{\xi}\partial^2\partial^\mu \langle 0|\, T(A_\mu(0)\psi_\alpha(y_1)\bar\psi_\beta(y_2))\,|0\rangle = e_B\left[\delta^{(4)}(x-y_1)\langle 0|\, T(\psi_\alpha(x)\bar\psi_\beta(y_2))\,|0\rangle - \delta^{(4)}(x-y_2)\langle 0|\, T(\psi_\alpha(y_1)\bar\psi_\beta(x))\,|0\rangle\right]. \tag{5}$$

Changing basis to momentum space, we have $\partial^\mu \longrightarrow iq^\mu$, where $q^\mu \equiv (k_2-k_1)^\mu$. We can set $x=0$ by applying $\int d^4y_1 d^4y_2\, e^{ik_1.y_1-ik_2.y_2}$ on both sides, giving

$$-\frac{i}{\xi}q^2 q^\mu D_{\mu\nu}(q)S_{\alpha\delta}(k_1)\Gamma^\nu_{\delta\lambda}(k_1,k_2)S_{\lambda\beta}(k_2) = e_B\left[S_{\alpha\beta}(k_2) - S_{\alpha\beta}(k_1)\right]. \tag{6}$$

In the last lecture, we showed $\frac{1}{\xi}\partial^2\partial^\mu D_{\mu\nu} = -k_\nu$. And so, the result, when written in terms of matrices in spinor space, reduces to

$$-S(k_1)(q_\nu\Gamma^\nu)S(k_2) = e_B\left(S(k_2) - S(k_1)\right), \tag{7}$$

or, equivalently,

$$q_\nu\Gamma^\nu(k_1,k_2) = e_B\left(S^{-1}(k_2) - S^{-1}(k_1)\right), \tag{8}$$

where $q \equiv k_2 - k_1$. This is an important constraint. To see the implications, we consider $k_1 = k$, k on-shell and $q \longrightarrow 0$, meaning k_2 is also close to on-shell. Then

$$\begin{aligned} S^{-1}(k_1) &\approx -\frac{1}{Z_2}(i\not{k}_1 + m - i\epsilon) + \dots \\ S^{-1}(k_2) &\approx -\frac{1}{Z_2}(i\not{k}_2 + m - i\epsilon) + \dots \end{aligned}$$

where m here is the physical mass. We then have

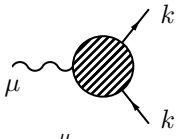
$$q_\nu \Gamma^\nu(k,k) = -\frac{e_B}{Z_2} i\not{q}, \tag{9}$$

or

$$\Gamma^\nu(k,k) = -\frac{ie_B}{Z_2}\gamma^\nu \tag{10}$$

when k is on shell. The physical charge we measure should be

$$\Gamma^\mu_{phys}(k,k) \quad = \quad$$



$$\equiv \quad -ie_{phys}\gamma^\mu. \tag{11}$$

where k is on-shell and we are using the physical fields. In other words, consider

$$G^{(phys)}_\mu(k_1,k_2) = \int d^4y_1 d^4y_2\, e^{ik_1.y_1 - ik_2.y_2} \langle 0|\, T(A_\mu(0)\psi_\alpha(y_1)\bar{\psi}_\beta(y_2))\,|0\rangle \tag{12}$$

where the fields are now the physical fields, rather than the bare fields. Then

$$G^{(phys)}_\mu(k_1,k_2) = D_{\mu\nu}(q)S(k_1)\Gamma^\nu_{phys}(k_1,k_2)S(k_2) \tag{13}$$

where $D_{\mu\nu}$ and S are again here for the physical fields. Since

$$A^B_\mu = \sqrt{Z_3}A_\mu, \quad D^B_{\mu\nu} = Z_3 D_{\mu\nu}, \quad \psi^B = \sqrt{Z_2}\psi, \quad S^B = Z_2 S, \tag{14}$$

we have that

$$G^B_\mu = \sqrt{Z_3}(\sqrt{Z_2})^2 G^{(phys)}_\mu, \tag{15}$$

where $G^B \equiv D^B S^B \Gamma^B S^B$ and $G^{(phys)} = DS\Gamma^{(phys)}S$. From this, we have that

$$\Gamma^\nu(k,k) = \sqrt{Z_3}Z_2\Gamma^\nu_B(k,k) \tag{16}$$

and so

$$e = \sqrt{Z_3}e_B. \tag{17}$$

The dependence of e on Z_2 cancels precisely as a result of $\Gamma_B \propto \frac{e_B}{Z_2}$. That $\frac{e}{e_B} = \sqrt{Z_3}$ only depends on Z_3, the field strength renormalization of the photon, has important implications: the ratio is universal for all charged fields. Suppose that $e^{proton}_B = e^{electron}_B$. Then it is necessarily true that $e^{proton} = e^{electron}$, despite the proton and electron interacting very differently and having different masses. If $\frac{e}{e_B}$ depended on Z_2, for example, then we would have an extremely difficult time in explaining why $e^{proton} = e^{electron}$, as their respective values of Z_2 are very different. Finally, in terms of renormalized quantities:

$$q_\nu \Gamma^\nu(k_1,k_2) = e\left[S^{-1}(k_2) - S^{-1}(k_1)\right]. \tag{18}$$

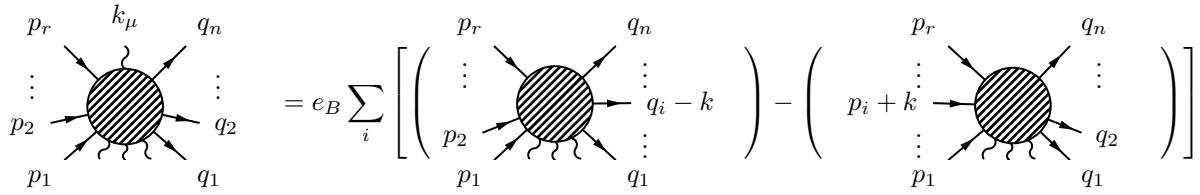
We note additionally that for k_1 and k_2 on-shell, but $k_1 \neq k_2$,

$$q_\nu \Gamma^\nu(k_1,k_2) = 0. \tag{19}$$

This is an example of a large class of identities. These are known as the general Ward identities. These identities are obtained by acting on the generating functional for connected diagrams with

$$\frac{\delta}{\delta J_{\nu_1}(z_1)}\frac{\delta}{\delta J_{\nu_2}(z_2)}\cdots\frac{\delta}{\delta\bar{\eta}(y_1)}\frac{\delta}{\delta\bar{\eta}(y_2)}\cdots\frac{\delta}{\delta\eta(x_1)}\frac{\delta}{\delta\eta(x_2)}\cdots, \tag{20}$$

and then setting $J_\mu = \bar{\eta} = \eta = 0$. The resulting expression is most transparently written diagramatically in momentum space:



(21)

where each external line should be considered as an exact photon or fermion propagator, except that associated with k_μ on the left-hand side, which should be amputated.

Remarks:

1. Suppose we attach to Γ^μ a photon propagator at the q^ν. If all the external propagators are on-shell, the longitudinal part of the propagator does not contribute as the inner product of this with q^ν is zero. This enforces gauge invariance.

2. Suppose we attach to Γ^μ an external photon line, that is, $\epsilon_\mu \Gamma^\mu(k, \ldots)$. This is invariant under $\epsilon_\mu \longrightarrow \epsilon_\mu + k_\mu$, which is a gauge transformation $A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda$.

3. Only charged particles need to be on-shell. Other photon lines or any other neutral particles (if they exist) can be off-shell, since they they do not transform under gauge transformations.

4. For Ward identities to be valid, regularization should preserve gauge invariance.

8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 14

We now consider the Lagrangian for quantum electrodynamics in terms of renormalized quantities.

$$\begin{aligned}\mathscr{L} &= -\frac{1}{4}F^B_{\mu\nu}F_B^{\mu\nu} - i\bar{\psi}_B(\gamma^\mu(\partial_\mu - ie_B A^B_\mu) - m_B)\psi_B \\ &= -\frac{1}{4}Z_3 F_{\mu\nu}F^{\mu\nu} - iZ_2\bar{\psi}(\gamma^\mu\partial_\mu - m - \delta m)\psi - Z_2 eA_\mu\bar{\psi}\gamma^\mu\psi.\end{aligned}$$

We know from previous lectures that there is no mass term for A_μ, that the bare and physical fields and couplings are related by

$$\begin{aligned}A^B_\mu &= \sqrt{Z_3}A_\mu, \\ \psi_B &= \sqrt{Z_2}\psi, \\ m_B &= m + \delta m, \\ e_B &= \frac{1}{\sqrt{Z_3}}e,\end{aligned}$$

and that there is no renormalization for the gauge fixing term. These results are a consequence of gauge symmetry, enforced through the Ward identities. We split the Lagrangian into three pieces:

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1 + \mathscr{L}_{ct}, \tag{1}$$

where we have

$$\begin{aligned}\mathscr{L}_0 &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\bar{\psi}(\gamma^\mu\partial_\mu - m)\psi, \\ \mathscr{L}_1 &= -eA_\mu\bar{\psi}\gamma^\mu\gamma, \\ \mathscr{L}_{ct} &= -\frac{1}{4}(Z_3 - 1)F_{\mu\nu}F^{\mu\nu} - i(Z_2 - 1)\bar{\psi}(\gamma^\mu\partial_\mu - m)\psi \\ &\quad -iZ_2\delta m\bar{\psi}\psi - (Z_2 - 1)eA_\mu\bar{\psi}\gamma^\mu\gamma.\end{aligned}$$

$\mathscr{L}_0$ is the free Lagrangian, $\mathscr{L}_1$ is the interaction Lagrangian, and $\mathscr{L}_{ct}$ is the counter-term Lagrangian. The parameters $Z_3 - 1$, $Z_2 - 1$ and δm are specified by the following renormalization conditions:

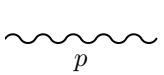
1. For the spinor propagator, $S(k) = \frac{1}{i\not{k} - m + i\epsilon - \Sigma(\not{k})}$,

$$\begin{aligned}\Sigma|_{\not{k}=-im} &= 0 \text{ (Physical mass condition)}, \\ \left.\frac{d\Sigma}{d\not{k}}\right|_{\not{k}=-im} &= 0 \text{ (Physical field condition)}.\end{aligned}$$

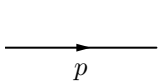
2. For the photon propagator, $D^T_{\mu\nu}(k) = \frac{P^T_{\mu\nu}}{k^2 - i\epsilon}\frac{1}{1-\Pi(k^2)}$,

$$\Pi|_{k=0} = 0 \text{ (Physical mass condition)}. \tag{2}$$

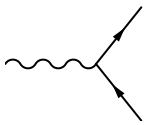
These three conditions allow us to fix our three parameters. We note that there is no need to introduce conditions on vertex corrections, and so, $\mathscr{L}$ is written in terms of physically measured masses and couplings. From this deconstruction, we acquire a set of Feynman rules for the interaction and counterterms in terms of the physical propagators.



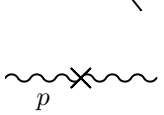
$$= \frac{-ig_{\mu\nu}}{k^2 + i\epsilon},$$



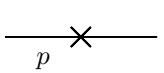
$$= \frac{1}{i\not{k} - m + i\epsilon},$$



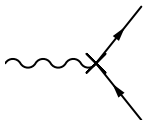
$$= -ie\gamma^\mu,$$



$$= -i(Z_3 - 1)(k^2 g^{\mu\nu} - k^\mu k^\nu) \sim O(e^2),$$



$$= -i(Z_2 - 1)(i\not{k} - m) + Z_2\delta m \sim O(e^2),$$

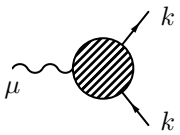


$$= -i(Z_2 - 1)e\gamma^\mu \sim O(e^3). \tag{3}$$

3.2: VERTEX FUNCTION

Consider the effective vertex we defined before:

$$\Gamma^\mu_{phys}(k,k) =$$



$$\equiv -ie_{phys}\gamma^\mu. \tag{4}$$

This is the physical vertex: it captures the full electromagnetic properties of a spinor interacting with a photon. As we showed in the previous lecture, the Ward identities impose that

$$\Gamma^\mu(k,k) = -ie\gamma^\mu \tag{5}$$

when k is on-shell, with $e = \frac{1}{\sqrt{Z_3}} e_B$ being the physical charge. We note that in this case, $q = 0$, and so this is an interaction with a static potential, measuring electric charge. We will now proceed to examine the general structure of $\Gamma^\mu(k_1, k_2)$, with k_1 and k_2 on-shell. We will discuss the physical interpretation, and we will compute the one-loop correction explicitly. For general $k_1^2 = k_2^2 = -m^2$, $q^2 = (k_2 - k_1)^2 \neq 0$, the process being described is an electron interacting a general external electromagnetic field. From Lorentz invariance, we can build Γ^μ from γ^μ, k_1^μ and k_2^μ. Hence,

$$i\Gamma^\mu(k_1, k_2) = \gamma^\mu A + i(k_2^\mu + k_1^\mu)B + (k_2^\mu - k_1^\mu)C, \tag{6}$$

where A, B, and C are 4×4 matrix functions of k_1 and k_2. But, since k_1 and k_2 are on-shell, and Γ^μ always appears in a product as

$$\bar{u}_{s'}(k_2)\Gamma^\mu(k_1, k_2)u_s(k_1) \tag{7}$$

where $u_{s'}(k_1)$ and $\bar{u}_s(k_2)$ are on-shell spinor wave functions, we can then simplify Γ^μ with the understanding that it will always be found in this combination, using the on-shell spinor identities

$$\begin{aligned} \not{k}u_s(k) &= -imu_s(k), \\ \bar{u}_s(k)\not{k} &= -im\bar{u}_s(k). \end{aligned}$$

Hence, A, B, and C are scalars, and functions of the scalars k_1^2, k_2^2 and $k_1.k_2$, or, equivalently, of q^2 and m. From the Ward identities, we have that

$$q_\mu \Gamma^\mu = 0, \tag{8}$$

and, as $\bar{u}_{s'}(k_2) \not{q} u_s(k_1) = 0$, and $q_\mu(k_1^\mu - k_2^\mu) = 0$, only the term in C on the left-hand side is non-zero. We therefore have $C = 0$. It is common to rewrite Γ^μ using the Gordon identity. Defining $\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$, this result states

$$\bar{u}_{s'}(k_2)\gamma^\mu u_s(k_1) = \frac{i}{2m}\bar{u}_{s'}(k_2)\left[(k_1^\mu + k_2^\mu) + iq_\nu \sigma^{\mu\nu}\right] u_s(k_1). \tag{9}$$

This allows us to exchange the term in B for a term in $\sigma^{\mu\nu}$.

Proof

$$\begin{aligned}
\bar{u}_{s'}(k_2)\gamma^\mu u_s(k_1) &= \frac{i}{2m}\left[\bar{u}_{s'}(k_2)\gamma^\mu \not{k}_1 u_s(k_1) + \bar{u}_{s'}(k_2)\not{k}_2 \gamma^\mu u_s(k_1)\right] \\
&= \frac{i}{2m}\left[\left(\frac{k_{2v}+k_{1v}}{2} - \frac{k_{2v}-k_{1v}}{2}\right)\bar{u}_{s'}(k_2)\gamma^\mu\gamma^\nu u_s(k_1)\right. \\
&\quad \left. + \left(\frac{k_{2v}+k_{1v}}{2} + \frac{k_{2v}-k_{1v}}{2}\right)\bar{u}_{s'}(k_2)\gamma^v\gamma^\mu u_s(k_1)\right] \\
&= \frac{i}{2m}\bar{u}_{s'}(k_2)\left[\left(\frac{k_{2v}+k_{1v}}{2}\right)\{\gamma^\mu,\gamma^\nu\} - \left(\frac{k_{2v}-k_{1v}}{2}\right)\left[\gamma^\mu,\gamma^\nu\right]\right] u_s(k_1) \\
&= \frac{i}{2m}\bar{u}_{s'}(k_2)\left[(k_2^\mu + k_1^\mu) + iq_v\sigma^{\mu\nu}\right] u_s(k_1).
\end{aligned}$$

□

From this, we find that

$$i\Gamma^\mu(k_1,k_2) = e\left[\gamma^\mu F_1(q^2) - \frac{\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\right]. \tag{10}$$

$F_1(q^2)$ and $F_2(q^2)$ are known as form factors. We have that $eF_1(q^2) = A + 2mB$, and $eF_2(q^2) = -2mB$. Note that the Ward identity means that $F_1(0) = 1$ exactly.

8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 15

3.3: ANOMALOUS MAGNETIC MOMENT

In the last lecture, we showed that the physical vertex $\Gamma^\mu(k_1,k_2)$ takes the general form

$$i\Gamma^\mu(k_1,k_2)=e\left[\gamma^\mu F_1(q^2)-\frac{\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\right], \tag{1}$$

and in the limit $k_1-k_2=q\longrightarrow 0$,

$$i\Gamma^\mu(k_1,k_2)\longrightarrow e\left[\gamma^\mu-\frac{\sigma^{\mu\nu}q_\nu}{2m}F_2(0)\right]\equiv\Gamma^\mu_{eff}(k_1,k_2). \tag{2}$$

This vertex is reproduced by the effective Lagrangian,

$$\mathscr{L}_{eff}=-i\bar\psi(\gamma^\mu\partial_\mu-m)\psi-eA_\mu\bar\psi\gamma^\mu\gamma-\frac{ieF_2(0)}{4m}\bar\psi\sigma^{\mu\nu}F_{\mu\nu}\psi. \tag{3}$$

Consider the case where ψ is non-relativistic, in a classical electromagnetic background A_μ. That is,

$$\begin{aligned} p^0\sim mv^2+m \qquad & \vec p\sim mv,\\ A^0\sim mv^2 \qquad & \vec A\sim mv,\end{aligned}$$

where $v\ll 1$. This is consistent, because $D^\mu=\partial^\mu-ieA^\mu=i(p-eA)^\mu$, so, A^0 and $\vec A$ interact with ψ and give it energy of the order mv^2, and momentum of the order mv. The Dirac equation now has the form

$$(\gamma^\mu(\partial_\mu-ieA_\mu)-m)\psi+\frac{eF_2(0)}{4m}F_{\mu\nu}\sigma^{\mu\nu}\psi=0, \tag{4}$$

or $i\partial_t\psi=H\psi$, with

$$H=m\beta+\vec\alpha.(\vec p-e\vec A)+eA^0+\frac{ieF_2(0)}{4m}\gamma^0\sigma^{\mu\nu}F_{\mu\nu}. \tag{5}$$

Here, $\beta=-i\gamma^0$ and $\alpha^i=-i\gamma^0\gamma^i$. We will choose the basis

$$\gamma^0=i\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix},\quad \gamma^i=i\begin{pmatrix} 0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}, \tag{6}$$

or, equivalently,

$$\beta=\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix},\quad \vec\alpha=\begin{pmatrix} 0 & \vec\sigma\\ -\vec\sigma & 0\end{pmatrix}. \tag{7}$$

From this, we find

$$\begin{aligned}\gamma^0\sigma^{0i} &= \frac{i}{2}\gamma^0\left[\gamma^0,\gamma^i\right]=\begin{pmatrix} 0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix},\\ \gamma^0\sigma^{ij} &= \frac{i}{2}\gamma^0\left[\gamma^i,\gamma^j\right]=-i\epsilon^{ijk}\begin{pmatrix}\sigma_k & 0\\ 0 & -\sigma_k\end{pmatrix}.\end{aligned}$$

We now write $\psi^T\equiv\begin{pmatrix}\phi & \chi\end{pmatrix}$, $F_{0i}\equiv E_i$ and $F_{ij}=\epsilon_{ijk}B_k$. The Dirac equation reduces to two coupled partial differential equations:

$$\begin{aligned} i\partial_t\phi &= m\phi+\vec\sigma.(\vec p-e\vec A)\chi+eA^0\phi+\frac{ieF_2(0)}{2m}\left[\vec\sigma.\vec E\chi-i\vec\sigma.\vec B\phi\right],\\ i\partial_t\chi &= -m\chi+\vec\sigma.(\vec p-e\vec A)\phi+eA^0\chi+\frac{ieF_2(0)}{2m}\left[-\vec\sigma.\vec E\phi+i\vec\sigma.\vec B\chi\right].\end{aligned}$$

We now let $\phi = e^{-imt}\Phi$, $\chi = e^{-imt}X$. As $i\partial_t\psi \sim \left[m + O(mv^2)\right]\psi$, Φ and X describe fluctuations with $\Delta E \sim mv^2$. In terms of these fields, taking the limit $v \longrightarrow 0$, the equations reduce to

$$\begin{aligned} i\partial_t\Phi &= \vec{\sigma}.\vec{\pi}X + eA^0\Phi + \frac{ieF_2(0)}{2m}\left[-i\vec{\sigma}.\vec{B}\phi\right] + O(v^3), \\ 0 &= -2mX + \vec{\sigma}.\vec{\pi}\Phi + O(v^2), \end{aligned}$$

where $\vec{\pi} = \vec{p} - e\vec{A}$, and so, solving the second equation for X, and inserting the result into the first equation, we obtain

$$\begin{aligned} X &= \frac{1}{2m}\vec{\sigma}.\vec{\pi}\Phi, \\ i\partial_t\Phi &= \frac{1}{2m}(\vec{\sigma}.\vec{\pi})^2\Phi + eA^0\Phi + \frac{eF_2(0)}{2m}\vec{\sigma}.\vec{B}\Phi. \end{aligned}$$

Now,

$$(\vec{\sigma}.\vec{\pi})^2 = \sigma_i\sigma_j\pi^i\pi^j = (\delta_{ij} + i\epsilon_{ijk}\sigma_k)\pi^i\pi^j = \pi^2 + e\vec{\sigma}.\vec{B}, \tag{8}$$

as $\left[\pi^i, \pi^j\right] = -ieF^{ij} = -ie\epsilon_{ijk}B_k$, and so we arrive at the following time-evolution equation in the limit $v \longrightarrow 0$:

$$i\partial_t\Phi = \left[\frac{1}{2m}(\vec{p} - e\vec{A})^2 + eA^0 + \frac{e}{2m}(1 + F_2(0))\vec{\sigma}.\vec{B}\right]\Phi. \tag{9}$$

We recognise the first two terms as the kinetic energy of a particle in an electromagnetic field and the electrostatic potential energy, respectively, and the third term takes the form of a magnetic interaction,

$$H_{mag} = -\vec{\mu}.\vec{B}, \tag{10}$$

with

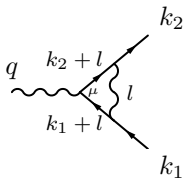
$$\vec{\mu} = -\frac{e}{2m}2(1 + F_2(0))\frac{\vec{\sigma}}{2} = \gamma\vec{S}, \tag{11}$$

with $\vec{S} = \frac{\vec{\sigma}}{2}$ the spin, and $\gamma = \frac{e}{2m}g$ the gyromagnetic ratio. Classically, we expect $g = 1$, and in the Dirac equation of quantum mechanics, we find $g = 2$. We see that in the case of quantum electrodynamics, we have $g = 2 + 2F_2(0)$. The additional term of $2F_2(0)$ is known as the anomalous magnetic moment. We will now explicitly compute the lowest order correction to the magnetic moment.

3.3.1: One-loop correction to the magnetic moment

To lowest order, the correction to the physical vertex function is given by

$\Gamma_1^\mu(k_1, k_2) \equiv$



$$= (-ie)^3\int\frac{d^4l}{(2\pi)^4}\gamma^\rho S_0(k_2 + l)\gamma^\mu S_0(k_1 + l)\gamma^\nu D^{(0)}_{\nu\rho}(l), \tag{12}$$

where $S_0(k) = \frac{-ik\!\!\!/ - m}{k^2 + m^2 - i\epsilon}$ and $D^{(0)}_{\mu\nu} = \frac{-ig_{\mu\nu}}{l^2 - i\epsilon}$. Explicitly, we have

$$\Gamma_1^\mu(k_1, k_2) = (-ie)^3(-1)^2(-i)\int\frac{d^4l}{(2\pi)^4}\frac{\gamma^\rho\left[i(k\!\!\!/_2 + l\!\!\!/) + m\right]\gamma^\mu\left[i(k\!\!\!/_1 + l\!\!\!/) + m\right]\gamma^\nu}{\left((k_1 + l)^2 + m^2 - i\epsilon\right)\left((k_2 + l)^2 + m^2 - i\epsilon\right)\left((l^2 - i\epsilon\right)}. \tag{13}$$

We can combine the denominators using the Feynman trick,

$$\frac{1}{A_1A_2A_3} = \int_0^1 dx_1\int_0^1 dx_2\int_0^1 dx_3\delta(x_1 + x_2 + x_3 - 1)\frac{2}{\left(x_1A_1 + x_2A_2 + x_3A_3\right)^3}, \tag{14}$$

reducing our result for Γ_1^μ to

$$\Gamma_1^\mu = 2e^3 \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \delta(x_1+x_2+x_3-1)\frac{N^\mu}{D^3}, \tag{15}$$

where

$$N^\mu \equiv \gamma^\nu \left[i(\not{k}_2+\not{l})+m\right]\gamma^\mu\left[i(\not{k}_1+\not{l})+m\right]\gamma_\nu, \tag{16}$$

and

$$\begin{aligned} D &\equiv x_1\left[(\not{k}_1+l)^2+m^2\right]+x_2\left[(\not{k}_2+l)^2+m^2\right]+x_3 l^2 - i\epsilon \\ &= (l+x_1k_1+x_2k_2)^2 + x_1(1-x_1)k_1^2 + x_2(1-x_2)k_2^2 - 2x_1x_2k_1.k_2 + (x_1+x_2)m^2 - i\epsilon. \end{aligned}$$

We may shift the variable in the integral $l \longrightarrow p = l + x_1k_1 + x_2k_2$, and rewrite the result in terms of q^2 instead of $k_1.k_2$, giving

$$D = p^2 + x_1x_2q^2 + (x_1+x_2)^2m^2 - i\epsilon. \tag{17}$$

Further,

$$\begin{aligned} N^\mu &= \gamma^\nu\left[i(\not{p}-x_1\not{k}_1+(1-x_2)\not{k}_2)+m\right]\gamma^\mu\left[i(\not{p}+(1-x_1)\not{k}_1+x_2\not{k}_2)+m\right]\gamma_\nu \\ &= -\gamma^\nu\not{p}\gamma^\mu\not{p}\gamma_\nu + \gamma^\nu\left[i(x_1\not{k}_1+(1-x_2)\not{k}_2)+m\right]\gamma^\mu\left[i((1-x_1)\not{k}_1+x_2\not{k}_2)+m\right]\gamma_\nu \\ &\quad +\text{terms linear in } p. \end{aligned}$$

The terms linear in p evaluate to zero in the integral, as they are odd, so we can discard them. The first term can be evaluated using the identity $\gamma^\nu\not{a}\not{b}\not{c}\gamma_\nu = -2\not{c}\not{b}\not{a}$, resulting in

$$-\gamma^\nu\not{p}\gamma^\mu\not{p}\gamma_\nu = -2\not{p}\not{p}\gamma^\mu + 4\not{p}p^\mu = -2p^2\gamma^\mu + 4p^\nu p^\mu\gamma_\nu. \tag{18}$$

This last term is again odd in the individual components of the momentum integral, and so reduces to $\frac{p^2 g^{\nu\mu}}{4}\gamma_\nu$ inside the integral. So, the contribution to the integrand from the first term is

$$-p^2\gamma^\mu. \tag{19}$$

We see that this term contributes to F_1, and we know by the Ward identity that $F_1(0)$ is zero. We can disregard this term here. The second term is $p-$independent, convergent in the ultraviolet, and contains a contribution to F_2. We use the identities

$$\begin{aligned} \gamma^\nu\gamma^\alpha\gamma^\beta\gamma_\nu &= 4g^{\alpha\beta}, \\ \gamma^\nu\gamma^\alpha\gamma_\nu &= -2\gamma^\alpha, \end{aligned}$$

and so the relevant contribution to N^μ is

$$\begin{aligned} N^\mu &= -2\left[-ix_2\not{k}_2+i(1-x_1)\not{k}_1\right]\gamma^\mu\left[-ix_1\not{k}_1+i(1-x_2)\not{k}_2\right] \\ &\quad +4m\left[i(1-2x_1)k_1^\mu+i(1-2x_2)k_2^\mu\right]-2m^2\gamma^\mu. \end{aligned}$$

We discard the last term, which again contributes to F_1. We again use the fact that Γ^μ appears in the combination $\bar{u}(k_2)\Gamma^\mu u(k_1)$ for on-shell spinors, and that $\bar{u}\not{k}_2 = -im\bar{u}$, $\not{k}_1 u = -imu$, for on-shell solutions. So, the relevant part of N^μ in the integrand can be written as

$$\begin{aligned} N^\mu &= -2\left[-x_2m+i(1-x_1)\not{k}_1\right]\gamma^\mu\left[-x_1m+i(1-x_2)\not{k}_2\right] \\ &\quad +4im\left[(1-2x_1)k_1^\mu+(1-2x_x)k_2^\mu\right]. \end{aligned}$$

Using the identity $\not{k}\gamma^\mu = -\gamma^\mu\not{k}+2k^\mu$, and again retaining only the parts contributing to F_2, the relevant part of N^μ reduces finally to

$$N^\mu = 2im(x_1+x_2)(1-x_1-x_2)(k_1^\mu+k_2^\mu). \tag{20}$$

Thus, we find

$$\Gamma_1^\mu = \gamma^\mu(\ldots) + (k_1^\mu+k_2^\mu)B, \tag{21}$$

with

$$B = 2e^3\int_0^1\int_0^1\int_0^1 dx_1dx_2dx_3\delta(x_1+x_2+x_3-1)\int\frac{d^4p}{(2\pi)^4}\frac{2im(x_1+x_2)(1-x_1-x_2)}{\left(p^2+x_1x_2q^2+(x_1+x_2)^2m^2-i\epsilon\right)^3}. \tag{22}$$

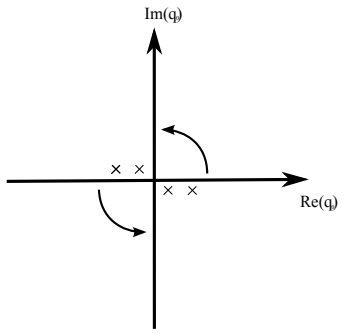


Figure 1: Illustration of the Wick rotation of the variable q_0.

Applying the Wick rotation, $p^0 \equiv ip_E^4$, $d^4p = id^4p_E$, we can explicitly evaluate

$$\int \frac{d^4p_E}{(2\pi)^4} \frac{1}{(p_E^2 + \Delta)} = \frac{1}{32\pi^2\Delta}, \tag{23}$$

and so

$$B = -4me^3 \frac{1}{32\pi^2} \int_0^1 \int_0^1 \int_0^1 dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 1) \frac{x_3(1-x_3)}{x_1x_2q^2 + (1-x_3)^2m^2}, \tag{24}$$

and, finally, for the form-factor F_2 we obtain the result

$$\begin{aligned} F_2(0) &= -\frac{2m}{e} B(q^2 = 0) = \frac{e^2}{4\pi} \int_0^1 dx_3 \int_0^{1-x_3} dx_2 \frac{x_3}{1-x_3} \\ &= \frac{e^2}{4\pi^2} \int_0^1 dx_3 x_3 = \frac{e^2}{8\pi^2} = \frac{\alpha}{2\pi}, \end{aligned}$$

where $\alpha \equiv \frac{e^2}{4\pi} \sim \frac{1}{137}$, and so

$$g = 2 + 2F_2(0) = 2 + \frac{\alpha}{\pi}. \tag{25}$$

$a_e = \frac{g-2}{2} = \frac{\alpha}{2\pi} = 0.0011614..$ at one-loop level. Experimentally, $a_e = 0.00115965218073(28)$ (Gabrielse 2008). Theoretically, the result is decomposed as

$$F_2(0) = \frac{\alpha}{2\pi} + a_2 \left(\frac{\alpha}{\pi}\right)^2 + a_3 \left(\frac{\alpha}{\pi}\right)^3 + a_4 \left(\frac{\alpha}{\pi}\right)^4, \tag{26}$$

where the second coefficient, a_2, consists of seven diagrams, and was calculated in 1957. The third coefficient consists of 72 diagrams, and was calculated in 1996. The fourth coefficient, a_4, consists of 891 diagrams.

8.324 Relativistic Quantum Field Theory II

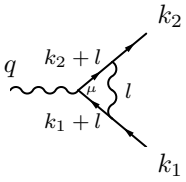
MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 16

Firstly, we summarize the results of the vertex correction from the previous lecture:

$$\Gamma_1^\mu(k_1,k_2) \equiv$$

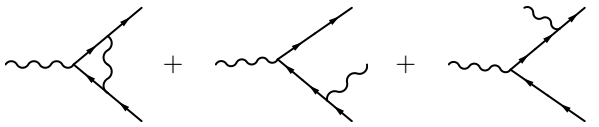


$$\begin{aligned} &= \gamma^\mu A(q^2) + i(k_1+k_2)^\mu B(q^2) \\ &= e\gamma^\mu F_1(q^2) - \frac{\sigma^{\mu\nu}q_\nu F_2(q^2)}{2m}, \end{aligned} \tag{1}$$

where $eF_1 = A + 2mB$ and $eF_2 = -2mB$. We showed that $F_2(0) = -\frac{2m}{e}B(0) = \frac{\alpha}{2\pi} = 0.0011614..$, where $\alpha \equiv \frac{e^2}{4\pi} \approx \frac{1}{137}$. The integral for Γ_1^μ is, in fact, infrared divergent. As $l \longrightarrow 0$,

$$\begin{aligned} (k_1+l)^2 + m^2 &= k_1^2 + m^2 + 2k_1.l + l^2 \\ &= 2k_1.l + l^2 \longrightarrow 0, \end{aligned}$$

and so $\Gamma_1 \sim \int d^4l \frac{1}{l^4}$ is divergent. This is due to soft photon interactions, and this effect is in fact cancelled by including the soft emissions:



(2)

The explanation is that it is only reasonable to calculate measurable cross-sections:

$$\left(\frac{d\sigma}{d\Omega}\right)_{measured} = \left(\frac{d\sigma}{d\Omega}\right)(\alpha \to \beta) + \left(\frac{d\sigma}{d\Omega}\right)(\alpha \to \beta + \text{soft photons}). \tag{3}$$

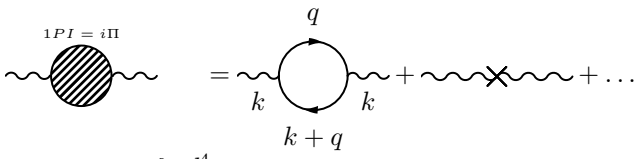
The calculation then proceeds by imposing an infrared cut-off λ on the photon momenta. The divergences in the $\lambda \longrightarrow 0$ limit cancel among virtual and real soft photon emissions, and we can safely take the $\lambda \longrightarrow 0$ limit in the end.

3.4: VACUUM POLARIZATION

We will now evaluate the one-loop correction to the photon propagator, and consider the physical interpretation, recalling the general structure we considered in lecture 12.

3.4.1: One-loop correction

$$i\Pi^{\mu\nu}(k) =$$



$$\begin{aligned} &= (-1)(-ie)^2 \int \frac{d^4q}{(2\pi)^4} \mathrm{tr}\left(\gamma^\mu S_0(k+q)\gamma^\nu S_0(q)\right) - i(Z_3-1)k^2 P_T^{\mu\nu} \end{aligned} \tag{4}$$

We note that the factor of (-1) at the front comes from the fermionic loop, that the trace is over the omitted spinor indices, and that S_0 is the electron propagator. Having now enough experience with one-loop diagrams, we will omit the details of the calculation and only emphasise the new aspects. Using $S_0(q) = \frac{1}{i\not{q}-m} = -\frac{i\not{q}+m}{q^2+m^2}$, we first introduce the Feynman parameters:

$$\Pi_1^{\mu\nu} = e^2 \int_0^1 dx \int \frac{d^4p}{(2\pi)^4} \frac{4N^{\mu\nu}}{(p^2+D)^2} + \text{counterterm}, \tag{5}$$

with $p \equiv q + xk$, $D \equiv x(1-x)k^2 + m^2 - i\epsilon$, and

$$\begin{aligned} 4N^{\mu\nu} &= \text{tr}\left[\gamma^\mu(i\not{k} + i\not{q} + m)\gamma^\nu(i\not{q} + m)\right] \\ &= \text{tr}\left[\gamma^\mu(i\not{p} + i(1-x)\not{k} + m)\gamma^\nu(i\not{p} - ix\not{k} + m)\right] \\ &= -\text{tr}\left[\gamma^\mu\not{p}\gamma^\nu\not{p}\right] + m^2\text{tr}\left[\gamma^\mu\gamma^\nu\right] + x(1-x)\text{tr}\left[\gamma^\mu\not{k}\gamma^\nu\not{k}\right] \\ &\quad + \text{terms linear in } p + \text{terms with an odd number of } \gamma\text{matrices}. \end{aligned}$$

We note that the trace of a term with an odd number of γ-matrices gives zero, and that

$$\begin{aligned} \text{tr}\left[\gamma^\mu\gamma^\nu\right] &= 4\eta^{\mu\nu}, \\ \text{tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] &= 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}). \end{aligned}$$

Hence, disregarding irrelevant terms, we can write

$$N^{\mu\nu} = -2p^\mu p^\nu + 2x(1-x)k^\mu k^\nu + (m^2 + p^2 - x(1-x)k^2)\eta^{\mu\nu}. \tag{6}$$

Secondly, we evaluate the integrals, extending to a general dimension d. We note that

$$\int \frac{d^dp}{(2\pi)^d} p^\mu p^\nu f(p^2) = \frac{\eta^{\mu\nu}}{d} \int \frac{d^dp}{(2\pi)^d} p^2 f(p^2), \tag{7}$$

and so

$$i\Pi^{\mu\nu}(k) = 4e^2 \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{A\eta^{\mu\nu} + p^2(1-\frac{2}{d})\eta^{\mu\nu} + 2x(1-x)k^\mu k^\nu}{(p^2+D)^2}, \tag{8}$$

where we have set $A \equiv m^2 - x(1-x)k^2$. We note that the first and third terms in the numerator are logarithmically divergent, and the second term is quadratically divergent. We now apply a Wick rotation $p^0 \to ip_E^d$, $d^dp \to id^dp_E$ and $p^2 \to p_E^2$. We recall that

$$\int \frac{d^dp_E}{(2\pi)^d} \frac{(p_E^2)^a}{(p_E^2+D)^b} = \frac{\Gamma(b-a-\frac{d}{2})\Gamma(a+\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(b)\Gamma(\frac{d}{2})} D^{-(b-a-\frac{d}{2})}, \tag{9}$$

and so

$$(1-\frac{2}{d}) \int \frac{d^dp_E}{(2\pi)^d} \frac{p_E^2}{(p_E^2+D)^2} = -D \int \frac{d^dp_E}{(2\pi)^d} \frac{1}{(p_E^2+D)^2}. \tag{10}$$

Hence, the numerator of (8) can be replaced by

$$(A-D)\eta^{\mu\nu} + 2x(1-x)k^\mu k^\nu. \tag{11}$$

The transverse component, therefore, is given by

$$-2x(1-x)k^2 P_T^{\mu\nu}, \tag{12}$$

and we can write

$$\begin{aligned} i\Pi^{\mu\nu}(k) &= -8ie^2k^2P_T^{\mu\nu}(k) \int_0^1 dx \int \frac{d^dp_E}{(2\pi)^d} \frac{1}{(p_E^2+D)^2} - i(Z_3-1)k^2P_T^{\mu\nu} \\ &= -8ie^2k^2P_T^{\mu\nu}(k) \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx \, \frac{x(1-x)}{D^{2-\frac{d}{2}}} - i(Z_3-1)k^2P_T^{\mu\nu}. \end{aligned}$$

Thirdly, we use dimensional regularization, setting $d = 4 - \epsilon$, $e \to e\mu^{\frac{\epsilon}{2}}$,

$$i\Pi^{\mu\nu}(k) = \frac{-8ie^2k^2P_T^{\mu\nu}(k)}{16\pi^2}\int_0^1 dx\, x(1-x)\left[\frac{2}{\epsilon} - \gamma + \log\left(\frac{4\pi\mu^2}{D}\right)\right] - i(Z_3-1)k^2P_T^{\mu\nu}, \tag{13}$$

and so

$$\Pi^{\mu\nu}(k) = k^2P_T^{\mu\nu}\Pi(k^2), \tag{14}$$

with

$$\Pi(k^2) = -\frac{e^2}{\pi^2}\int_0^1 dx\, x(1-x)\frac{1}{\epsilon} - \frac{1}{2}\log\left(\frac{D}{4\pi\mu^2e^{-\gamma}}\right) - (Z_3-1). \tag{15}$$

The physical field condition constrains that $\Pi(k^2=0) = 0$, and so Z_3 is fixed by

$$Z_3 = 1 - \frac{e^2}{6\pi^2}\frac{1}{\epsilon} - \frac{1}{2}\log\left(\frac{m^2}{4\pi\mu^2e^{-\gamma}}\right) \tag{16}$$

and the final result for $\Pi(k^2)$ is

$$\Pi(k^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\log\left(1 + \frac{k^2x(1-x)}{m^2}\right) \tag{17}$$

Remarks:

1. For $k^2 < -4m^2$, $\Pi(k^2)$ becomes complex.

2. For more than one charged particle, we must add their respective contributions, and the smallest m contributes most.

3. The internal propagator is given by

$$\begin{aligned} e^2D^{\mu\nu}(q) &\to \frac{-ie^2}{q^2-i\epsilon}\frac{\eta^{\mu\nu}}{1-\Pi(q^2)} \\ &= \frac{-ie^2\eta^{\mu\nu}}{q^2-i\epsilon}(1 + \Pi(q^2) + \ldots). \end{aligned}$$

We see that for $q^2 \gg m^2$, a large spacelike momentum, from (17) we have

$$\begin{aligned} \Pi(q^2) &\approx \frac{e^2}{2\pi^2}\log\frac{q^2}{m^2}\int_0^1 dx\, x(1-x) \\ &= \frac{\alpha}{3\pi}\log\frac{q^2}{m^2}, \end{aligned}$$

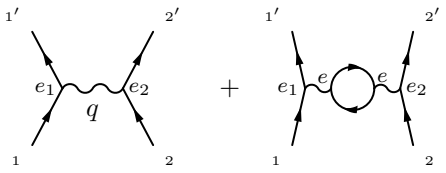
where $\alpha = \frac{e^2}{4\pi}$. Then, the internal propagator goes as

$$\frac{e^2}{q^2-i\epsilon} \longrightarrow \frac{e^2}{1-\Pi(q^2)}\frac{1}{q^2-i\epsilon} \equiv \frac{e^2(q)}{q^2-i\epsilon}, \tag{18}$$

with $e^2(q) = \frac{e^2}{1-\frac{\alpha}{3\pi}\log\frac{q^2}{m^2}}$ the running coupling constant.

3.4.2: Physical implication

Consider scattering of two charged particles with coupling constants e_1 and e_2, for example, in the process $e^- + \mu^- \longrightarrow e^- + \mu^-$:



(19)

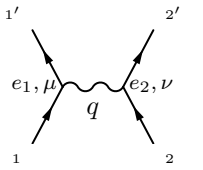
to lowest order, where e is the coupling constant associated with the lightest intermediate particle.

$$S(1,2 \longrightarrow 1',2') \quad = \quad (-ie)^2 \bar{u}_{1'}(p_1')\gamma^\mu u_1(p_1) D_{\mu\nu}(q) \bar{u}_{2'}(p_2')\gamma^\nu u_2(p_2),$$

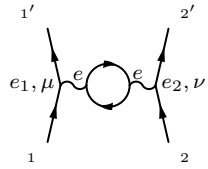
where

$$D_{\mu\nu}(q) = \frac{-i}{q^2 - i\epsilon}\frac{P^T_{\mu\nu}(q)}{1-\Pi(q^2)} + D^L_{\mu\nu}(q), \tag{20}$$

and $P^T_{\mu\nu} = \eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}$, $q = p_1' - p_1 = p_2' - p_2$. Note that $\bar{u}_{2'}(p_2')\gamma^\nu u_2(p_2) q_\nu = \bar{u}_{2'}(p_2')(\not{p}_2' - \not{p}_2)u_2(p_2) = 0$. We now want to consider corrections to the Coulomb potential. We consider the low energy, non-relativistic limit, where we derive most of our intuition about electromagnetism from, and where the notion of a potential makes most sense. The lowest two diagrams drawn above correspond to



$$= \quad \frac{e_1 e_2}{q^2}\eta_{\mu\nu},$$



$$= \quad \frac{e_1 e_2}{q^2}\Pi(q^2)\eta_{\mu\nu}. \tag{21}$$

In the non-relativistic limit,

$$\left|q^0\right| \sim |v\vec{q}| \ll |\vec{q}|\,, \quad q^2 \approx \vec{q}^2. \tag{22}$$

In this case, one-photon exchange corresponds to the Born approximation:

$$\frac{e_1 e_2}{\vec{q}^2} = \int d^3\vec{r}\, e^{-i\vec{q}.\vec{r}} V_0(\vec{r}), \tag{23}$$

where $V_0(\vec{r}) = \frac{e_1 e_2}{4\pi|\vec{r}|}$ is the Coulomb potential. The one-loop correction is given by $\frac{e_1 e_2}{\vec{q}^2}\Pi(\vec{q}^2)$, where, from (17)

$$\Pi(\vec{q}^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\log\left(1 + \frac{\vec{q}^2}{m^2}x(1-x)\right), \tag{24}$$

and the term provides a correction to the Coulomb potential

$$\delta V(\vec{r}) = e_1 e_2 \int d^3\vec{r}\, e^{-i\vec{q}.\vec{r}}\frac{\Pi(\vec{q}^2)}{\vec{q}^2}. \tag{25}$$

From now on, we will for convenience write $q \equiv |\vec{q}|$, $r \equiv \|\vec{r}\|$. The angular integral is given by

$$\begin{aligned} & \frac{2\pi}{(2\pi)^3}\int_0^\pi d\theta \sin\theta e^{iqr\cos\theta} \\ = \quad & \frac{1}{4\pi^2}\frac{1}{iqr}\left(e^{iqr} - e^{-iqr}\right), \end{aligned}$$

and so we find for the correction to the Coulomb potential

$$\begin{aligned} \delta V(\vec{r}) \quad &= \quad \frac{e_1 e_2}{4\pi^2}\int_0^\infty dq\, \frac{1}{iqr}\left(e^{iqr} - e^{-iqr}\right)\Pi(\vec{q}^2) \\ &= \quad \frac{e_1 e_2}{4\pi^2}\int_{-\infty}^\infty dq\, \frac{1}{iqr}e^{iqr}\Pi(\vec{q}^2). \end{aligned}$$

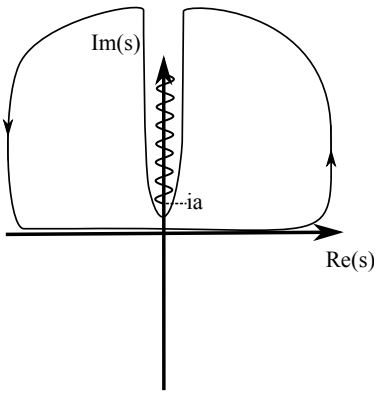


Figure 1: Clockwise contour for the integral in dz in (26). The semi-circular arc is taken to infinity and gives a vanishing contribution.

If we now let $z \equiv qr$, $a \equiv \frac{mr}{\sqrt{x(1-x)}} \geq 2mr$, the result reduces to

$$\delta V(\vec{r}) = \frac{e_1 e_2}{8\pi^4} \frac{e^2}{ir} \int_0^1 dx\, x(1-x) \int_{-\infty}^{\infty} dz\, \frac{e^{iz}}{z} \log\left(1 + \frac{z^2}{a^2}\right). \tag{26}$$

The integral in dz, $I = \int_{-\infty}^{\infty} dz\, \frac{e^{iz}}{z} \log\left(1 + \frac{z^2}{a^2}\right)$, can be computed using the complex contour shown in figure 1, giving

$$\begin{aligned} I &= 2\int_a^{\infty} d\lambda \frac{e^{-\lambda}}{\lambda} i\pi \\ &= 2i\pi \int_1^{\infty} d\lambda\, \frac{e^{-a\lambda}}{\lambda} \end{aligned}$$

after setting $\lambda \to a\lambda$. So, our result for the correction to the Coulomb potential is given by

$$\begin{aligned} \delta V(\vec{r}) &= \frac{e_1 e_2}{4\pi} \frac{e^2}{\pi^2} \int_0^1 dx\, x(1-x) \int_1^{\infty} \frac{d\lambda}{\lambda}\, e^{-\lambda \frac{mr}{\sqrt{x(1-x)}}} \\ &\equiv \frac{e_1 e_2}{4\pi} Z(mr) \end{aligned}$$

Remarks:

1. When $mr \gg 1$, we can evaluate the integral in the saddle-point approximation, using integration by parts,

$$\delta V(r) = \frac{e_1 e_2}{4\pi} \frac{e^2}{16\pi^{\frac{3}{2}}} \frac{e^{-2mr}}{(mr)^{\frac{3}{2}}} + \ldots. \tag{27}$$

2. $Z(mr)$ increases with decreasing r. As $r \longrightarrow 0$, $I(r) \longrightarrow \infty$. If we consider putting a short-distance cut off at $mr = e \ll 1$, we find

$$Z(\epsilon) = \frac{e^2}{6\pi^2} \log \frac{1}{\epsilon} + \ldots, \tag{28}$$

and thus

$$\begin{aligned} V(r) &= V_0 + \delta V(r) \\ &= \frac{e_1 e_2}{4\pi r}(1 + Z(mr)) \\ &= \frac{\tilde{e}_1(r)\tilde{e}_2(r)}{4\pi r}, \end{aligned}$$

where

$$\tilde{e}_i(r) = e_i \left(1 + Z(mr)\right)^{\frac{1}{2}}. \tag{29}$$

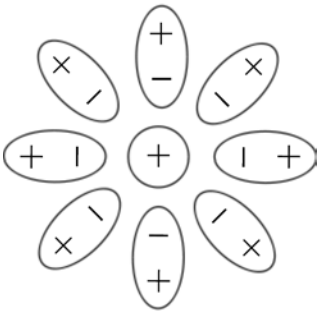


Figure 2: Virtual electron-positron pairs form a screening effect.

We observe that $\tilde{e}_i(r) \longrightarrow \infty$ as $r \longrightarrow 0$, and $\tilde{e}_i(r) \longrightarrow e_i$ as $r \longrightarrow 0$, with small experimental corrections. Physically, we can view $\tilde{e}_i(r = \epsilon)$ as the bare charge, which is very large. The physical interpretation is that virtual electron-positron pairs screen the charge more at large distances. The screening length is given by $r_s \sim \frac{1}{2m}$. That is, there is a negative cloud of size $\frac{1}{2m}$. At large distances, the charges scale as $e_i \sim e^{-2mr}$.

8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 17

4: GENERAL RENORMALIZATION THEORY

Let us recall some of the major results and methods of previous lectures:

1. In perturbation theory, bare and physical quantities are related by ultraviolet-divergent expressions:

$$m_{phys} = m_B + \delta m, \tag{1}$$

where m_{phys} is finite, δm is ultraviolet-divergent, and so m_B is necessarily ultraviolet-divergent.

2. We express the Lagrangian in terms of physical quantites, and separate it into

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_I + \mathscr{L}_{ct}, \tag{2}$$

where $\mathscr{L}_0$ is the canonically normalized free Lagrangian for physical fields and masses, $\mathscr{L}_I$ contains the interaction, again in terms of physical parameters, and $\mathscr{L}_{ct}$ contains the counterterms with ultraviolet-divergent coefficients. From $\mathscr{L}_0$, we obtain the propagators of the physical fields. $\mathscr{L}_I$ and $\mathscr{L}_{ct}$ give interaction vertices.

3. At the one-loop level, the self-energy is given by the effective two-point vertices: the 1PI two-point vertex of the interaction and the counter-term two-point vertex. The counterterms absorb ultraviolet divergences, and the finite parts of the counterterms are determined by renormalization conditions, which ensure the quantities in $\mathscr{L}_0 + \mathscr{L}_I$ are physical. The conditions constrain the self-energy and the effective vertices, and give a finite, uniquely-determined value for the counterterms.

There are several questions we need to consider:

1. Can counterterms remove higher order divergences in the self-energies and vertex corrections?
2. Can they remove ultraviolet divergences in generic physical observables?
3. How does the procedure work for a general theory?
4. What is the physics behind the success (or failure) of renormalization of ultraviolet divergences?

4.1: DEGREES OF DIVERGENCES

Given a generic 1PI diagram M in quantum electrodynamics, or any theory, how can we tell whether it is ultraviolet divergent or not? We begin by introducing the superficial degree of divergence, D. This is defined by

$$\begin{aligned} D \;=\; & \text{number of factors of internal momentum in the numerator} \;- \\ & \text{number of factors of internal momentum in the denominator} \end{aligned}$$

If we let all the loop momenta go to infinity with a common factor $S \longrightarrow \infty$, then

$$M \sim \int^{\Lambda} dS\, S^{D-1} \sim \begin{cases} \Lambda^D & D > 0, \\ \log \Lambda & D = 0, \\ \text{finite} & D < 0. \end{cases} \tag{3}$$

It is the superficial degree because it gives a rough indication of the behaviour. We will mention caveats and how to deal with them later in the lecture. We will now find an explicit expression for D in the case of quantum electrodynamics. Introduce

$$\begin{aligned} E_e &\equiv \text{Number of external electron lines,} \\ E_\gamma &\equiv \text{Number of external photon lines,} \\ I_e &\equiv \text{Number of internal electron lines,} \\ I_\gamma &\equiv \text{Number of internal photon lines,} \\ V &\equiv \text{Number of vertices,} \\ L &\equiv \text{Number of loops.} \end{aligned}$$

Then we have

$$D = 4L - I_e - 2I_\gamma. \tag{4}$$

We can express I_e, I_γ and L in terms of E_e, E_γ and V:

$$\begin{aligned} 2I_\gamma + E_\gamma &= V, \\ 2I_e + E_e &= 2V, \\ I_e + I_\gamma - (V-1) &= L, \end{aligned}$$

and so $I_\gamma = \frac{1}{2}(V - E_\gamma)$, $I_e = V - \frac{E_e}{2}$ and we find

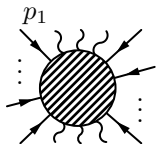
$$D = 4 - E_\gamma - \frac{3}{2}E_e. \tag{5}$$

We see that D only depends on the number of external legs, not on the internal structure of a diagram. In order to have ultraviolet divergences, we require $D \geq 0$, and so

$$E_\gamma + \frac{3}{2}E_e \leq 4. \tag{6}$$

Therefore, only a finite number of external lines can yield superficially divergent integrals. So we have a finite number of classes of divergent diagrams. They precisely correspond to the counter terms we discussed earlier. We will enumerate them in the next section. Let us now generalize (4) to a general theory. First, we provide an alternative derivation:

$$\begin{aligned} M\delta(p_1 + \ldots) &= \\ &= \text{Fourier transform of } \left\langle \psi(x_1)\ldots\psi(x_{E_e})A(y_1)\ldots A(y_{E_\gamma})\right\rangle \\ &\quad \text{with external legs amputated.} \end{aligned} \tag{7}$$



Hence,

$$M \sim \Lambda^D e^V \tag{8}$$

where e is the coupling constant. We have that $[M] = D$, as $[e] = 0$, and therefore

$$\begin{aligned} [M] - 4 &= \left[\frac{\int d^4x_1 \ldots d^4y_1 \ldots \langle\psi(x_1)\ldots A(y_1)\ldots\rangle}{\int d^4x_1 \left\langle\bar{\psi}(x_1)\psi(x_1)\right\rangle \ldots \int d^4y_1 \left\langle A(y_1)A(y_1)\right\rangle \ldots}\right] \\ &= -E_e\left[\psi\right] - E_\gamma\left[A\right]. \end{aligned}$$

We note that $[\psi] = \frac{3}{2}$ and $[A] = 1$, and hence

$$D = [M] = 4 - \frac{3}{2}E_e - E_\gamma. \tag{9}$$

Now, for a general theory in d spacetime dimensions, the field content is given by ϕ_f, $f = 1, 2, \ldots$, where f labels the field type. $[\phi_f] = \Delta_f$ and $\Delta_f \geq 0$ in all physical theories. We have interaction vertices of type i, $i = 1, 2, \ldots$, contributing a term of the form

$$\lambda_i(\partial)^{n_i} \prod_f \phi_f^{n_{if}}. \tag{10}$$

Here, λ_i is the coupling constant, with dimension

$$[\lambda_i] = \delta_i = d - n_i - \sum_f n_{if}\Delta_f. \tag{11}$$

Now consider a 1PI diagram in such a theory:

$$\begin{aligned} E_f &\equiv \text{number of external lines of } \phi_f, \\ V_i &\equiv \text{number of vertices of type } i. \end{aligned}$$

Then $M \sim \Lambda^D \prod_i \lambda_i^{V_i}$ and so $D = [M] - \sum_i V_i\delta_i$. Again,

$$[M] - d = \sum_f E_f\Delta_f \tag{12}$$

and so the general expression for the superficial degree of divergence is given by

$$D = d - \sum_f E_f\Delta_f - \sum_i V_i\delta_i. \tag{13}$$

Diagrams which are ultraviolet divergent satisfy

$$\sum_f E_f\Delta_f + \sum_i V_i\delta_i \leq d. \tag{14}$$

We note that $\Delta_f \geq 0$ in all physical theories. We can now divide all theories into

1. All $\delta_i > 0$: there are only a finite number of superficially divergent diagrams. These are **super-renormalizable** theories.

2. All $\delta_i \geq 0$: there are a finite number of classes of divergent diagrams. These are **renormalizable** theories.

3. There exists at least one $\delta_i < 0$. In these theories, increasing V_i means increaing D, so all amplitudes are divergent at high enough orders. These are **non-renormalizable** theories.

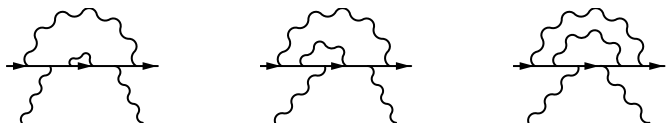
These terms also apply to individual interactions for a vertex of type i:

1. $\delta_i > 0$: super-renormalizable, **relevant** interaction.

2. $\delta_i = 0$: renormalizable, **marginal** interaction.

3. $\delta_i < 0$: non-renormalizable, **irrelevant** interaction.

Note that a diagram can be superficially convergent but divergent because of a divergent supdiagram. For example, in quantum electrodynamics, in the case $E_e = 2$, $E_\gamma = 2$, in which we have that $D = -1$



(15)

Figure 1: The first two diagrams with $E_e = 2$, $E_\gamma = 2$ both diverge because they contain divergent subdiagrams, whereas the third diagram is finite.

8.324 Relativistic Quantum Field Theory II

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Hong Liu, Fall 2010

Lecture 18

4.2: CANCELLATION OF DIVERGENCES

4.2.1: General Structure

Consider a generic divergent diagram M of degree D; that is,

$$M \sim \int^{\Lambda} ds\, s^{D-1} \tag{1}$$

if all loop momenta are taken proportional to s. Generically, internal propagators have the form

$$\frac{1}{(as+p)^{\alpha}+\ldots} \sim \frac{1}{s^{\alpha}} \quad \alpha = \begin{cases} 2 & \text{scalar or vector,} \\ 1 & \text{fermion.} \end{cases} \tag{2}$$

for large s, where a is a numerical constant and p is a combination of the external momenta. Differentiating M, with respect to p, n times gives a term proportional to

$$\frac{1}{(as+p)^{\alpha+n}} \sim \frac{1}{s^{\alpha+n}}, \tag{3}$$

and so $D+1$ derivatives with respect to the external momenta will make M finite. This means that

$$M(p) = M_0 + M_1 p + \ldots M_D p^D + \text{finite terms}, \tag{4}$$

where the argument p of the function represents the collection of external momenta, we have suppressed the index structure, and M_0,$M_1, \ldots, M_D$ are potentially divergent constants. Suppose that M has E_f external lines of the field ϕ_f. Then divergences of $M(p)$ can be cancelled by counterterms of the form

$$\sum_{j=0}^{D} A_j (\partial)^j \prod_f \phi_f^{E_f} \tag{5}$$

where the A_j are divergent coefficients in order to cancel divergences in M_j. The index structure in $A_j \partial^j$ should match the suppressed index structure of $M(p)$.

4.2.2: Renormalization of Quantum Electrodynamics

In the case of quantum electrodynamics,

$$D = 4 - E_\gamma - \frac{3}{2} E_e, \tag{6}$$

and hence, ultraviolet-divergent diagrams are given by

$$E_\gamma + \frac{3}{2} E_e \leq 4. \tag{7}$$

We now enumerate these classes of diagrams:

1. $E_\gamma = E_e = 0$, $D = 4$: These are the vacuum diagrams, and give a divergent constant Λ_0. The divergence can be removed by including a constant Λ_{ct} counterterm in the Lagrangian, such that

$$\Lambda_{phys} = \Lambda_0 - \Lambda_{ct}. \tag{8}$$

Λ_{phys} does not have any physical significance, unless we are dealing with a theory of gravity. Λ_{phys} is measured to be very small, and it acts as a cosmological constant.

2. $E_e = 1$, $E_\gamma = 0, 1, 2$ and $E_e = 0$, $E_\gamma = 1$: These terms are all zero because of Lorentz invariance.

3. $E_e = 0$, $E_\gamma = 2$, $D = 2$: This is the photon self-energy,

$$\Pi^{\mu\nu}(q) = \quad \mu \quad q \quad \nu \quad . \tag{9}$$

Expanding this term, we get

$$\Pi^{\mu\nu}(q) = c_1\eta^{\mu\nu} + c_2 q^2\eta^{\mu\nu} + c_3 q^\mu q^\nu + \text{finite terms}. \tag{10}$$

The Ward identity imposes

$$\Pi^{\mu\nu}(q) = q^2 P_T^{\mu\nu}(q)\Pi(q^2), \tag{11}$$

and so we have

$$\Pi(q^2) = C + \text{finite terms} \tag{12}$$

where C is the only divergent constant. The divergence in C can be absorbed in the counterterm vertex,

$$\underset{p}{\sim\!\times\!\sim} = (Z_3 - 1)F^{\mu\nu}F_{\mu\nu}. \tag{13}$$

4. $E_e = 0$, $E_\gamma = 3$, $D = 1$: This is zero by charge conjugation invariance, as will be shown on the problem set.

5. $E_e = 0$, $E_\gamma = 4$, $D = 0$: This is the four-point vertex,

$$M_{\mu\nu\rho\sigma} \equiv \quad \begin{matrix} \mu & & \nu \\ & \bigcirc & \\ \sigma & & \rho \end{matrix}$$
$$= K(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) + \text{finite terms}, \tag{14}$$

where K is a potentially divergent constant, and the form of the divergent term follows from symmetry. The Ward identity constrains that

$$q^\mu M_{\mu\nu\rho\sigma} = 0, \tag{15}$$

and so $K = 0$, and the four-point vertex is convergent.

6. $E_e = 2$, $E_\gamma = 0$, $D = 1$: This is the fermion self-energy,

$$\Sigma(k) \equiv \quad \rightarrow\!\bigcirc\!\rightarrow$$
$$= A + B\not{k} + \text{finite terms}. \tag{16}$$

Here, A and B are divergent constants, and they can be cancelled by the counterterm vertex,

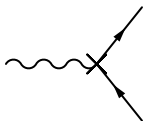
$$\underset{k}{\rightarrow\!\times\!\rightarrow} = (Z_2 - 1)(i\not{k} - m) + Z_2\delta m, \tag{17}$$

with the mass renormalization δm and the field strength renormalization Z_2 chosen appropriately.

7. $E_e = 2$, $E_\gamma = 1$, $D = 0$: This is the vertex function,

$$\Gamma^\mu(k, k') = A\gamma^\mu + \text{finite terms}. \tag{18}$$

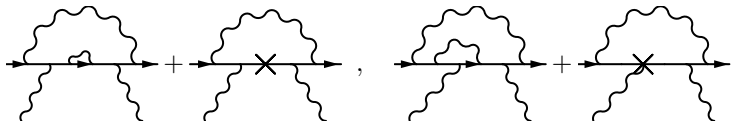
where A is a logarithmically divergent coefficient. This can be cancelled by the counterterm three-point vertex,



$$= -Z_1 e A_\mu \bar{\psi} \gamma^\mu \psi, \tag{19}$$

where the Ward identity imposes $Z_1 = Z_2$.

In summary, the counter terms we considered before are enough to cancel all of the superficial divergences to all orders in perturbation theory. We still need to consider diagrams with divergent sub-diagrams. Generically, these can be easily cancelled since divergent sub-graphs must also belong to those considered above. For example, the pairings



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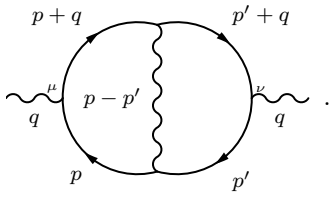
are both finite. We now need to consider the case of overlapping divergences. That is, when two divergent subgraphs share an internal line. These divergences are more difficult to deal with. For example, consider the diagram



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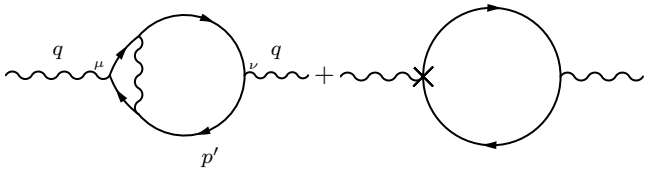
In the case that p is taken to be large, the pairing



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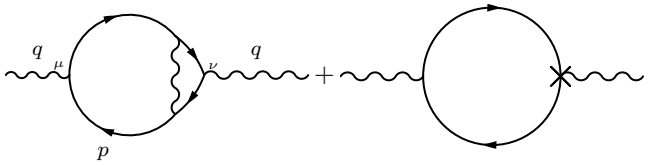
gives a vertex correction at the vertex μ, leaving a residual self-energy divergence for the case $p' \longrightarrow \infty$. Similarly, if p' is taken to be large, the pairing

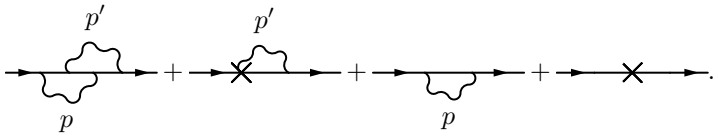


(23)

gives a vertex correction at the vertex ν, leaving a residual self-energy divergence for the case $p' \longrightarrow \infty$. The two procedures are not independent, and it is finally necessary to add a correction of the form

(24)

at the order $O(e^4)$ to cancel the divergences when both p and p' are large. Another example is given by the combination of diagrams



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It can be proved to all orders, the content of the Bogoliubov-Parasiuk-Hepp-Zimermann theorem, that overlapping divergences can be canceled by counterterms corresponding to superficial divergences.

8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 19

The procedure described for quantum electrodynamics last lecture can be readily generalized to other theories:

$$D = d - \sum_f E_f \Delta_f - \sum_i V_i \delta_i, \tag{1}$$

with $\Delta_f > 0$. For a renormalizable theory, all $\delta_i \geq 0$. That is, all the interactions are renormalizable. This means only for a finite set of classes of diagrams characterized by $\{E_f\}$ is $D \geq 0$, and D is always bounded by d. Therefore, all superficial divergences can be canceled be a finite set of counterterms of the form

$$\sum_{j=0}^{D} A_j \partial^j \prod_f \phi_f^{E_f} \tag{2}$$

where the A_j are divergents constants. From the facts that $\Delta_f > 0$ and $D \geq 0$, this set of interactions are all renormalizable: $[A_i] \geq 0$. Therefore, all divergences can be absorbed into redefinitions of the couplings, masses, and fields, provided the Lagrangian contains all the renormalizable interactions allowed by symmetries, in order to make sure the overlapping divergences cancel.

Remarks:

1. For quantum electrodynamics, there is no need to enumerate all the diagrams. We only need to write down all possible counterterms consistent with gauge symmetry.
2. Non-renormalizable theories, with some $\delta_i < 0$, require an infinite number of counterterms.
3. In a super-renormalizable theory, each counterterm contains only a finite number of divergent terms, when expanded in perturbation theory.
4. Generally any divergence allowed by symmetries will occur.
5. If the regulator for ultraviolet divergences does not violate a given symmetry, the divergences violating that symmetry will not occur. This is particularly important and subtle for gauge symmetries, because of the need of gauge fixing. For quantum electrodynamics, the Ward identities ensure this. For non-Abelian gauge theories, the generalized Ward identities arising from the BRST symmetry ensure this.
6. There are situations in which one cannot find any regulators which preserve a symmetry, and so the symmetry can be violated when the regulator is removed. This leads to the concept of an anomaly: a symmetry of the classical Lagrangian which is not respected in quantum theory. If this happens to a gauge symmetry, it leads to an inconsistency.

4.2.3: Ultraviolet Divergences and Renormalization in Coordinate Space

Our discussion has been so far restricted to momentum space, with loop momenta as $p \longrightarrow \infty$, leading to ultraviolet divergences. One would normally expect from that uncertainty principle that ultraviolet divergences are associated with short-distance phenomena. This can be made manifest by calculating the Feynman diagrams in coordinate space, leading to the result that counterterms must be local. This also leads to the result that in systems at a finite temperature T, the counterterms are independent of T. We will not give a complete treatment, but we will briefly highlight the local nature of the counterterms. We will use a scalar field for illustration. In the free theory,

$$\mathscr{L}_0 = -\frac{1}{2} (\partial\phi)^2 - \frac{1}{2} m^2 \phi^2. \tag{3}$$

If we try to compute operators like $\phi^2(x)$, $\phi^3(x)$, etc., we find that they are not well-defined, containing ultraviolet divergences. For example, consider

$$\phi(x)\phi(y) = G(x-y) + :\phi(x)\phi(y): \tag{4}$$

where the first term is the contraction and the second term is the normal ordering. As $x \longrightarrow y$, the first term is singular: $G(x-y) \sim \frac{1}{(x-y)^{d-2}}$. The second term is regular in this limit. Hence,

$$\phi^2(x) = C_0 1 + :\phi^2(x):, \tag{5}$$

where $C_0 \sim \frac{1}{\epsilon^{d-2}}$, for a short-distance regulator ϵ. Similarly,

$$\phi^4(x) = \tilde{C}_0 1 + C_1 :\phi^2(x): + :\phi^4(x): . \tag{6}$$

In the interacting theory, we consider, for example,

$$\mathscr{L}_{int} = \lambda\phi^4, \tag{7}$$

or more generally

$$\mathscr{L}_{int} = \lambda O, \tag{8}$$

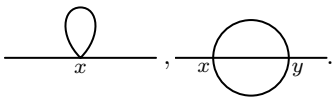
where O is some composite operator built from ϕ. The general physical amplitude is given, in the interaction picture, by

$$\langle T(X)\rangle = \left\langle T(X e^{i\int d^d x\, \lambda O(x)})\right\rangle_0 \tag{9}$$

where X is a collection of fields (operators), and the right-hand side is evaluated in the free theory. At the nth order in perturbation theory, the contribution is

$$\lambda^n \int \prod_{i=1}^{n} d^d x_i \ \langle T(X O(x_1)\dots O(x_n)\rangle_0 . \tag{10}$$

There are two types of divergent loops. Considering, for example, $O = \phi^4$:



(11)

The first type corresponds to a single propagator attached to the same vertex, and the second case corresponds to $O(x)O(y)|_{x\to y}$. The first type of divergence can be takencare of by normal ordering O itself:

$$\phi^4(x) = \tilde{C}_0 1 + C_1 :\phi^2(x): + :\phi^4(x): . \tag{12}$$

Here, the first term is canceled by Λ_0, and the second term is cancelled by a mass renormalization. The second type of divergence can also be dealt with using normal ordering:

$$:\phi^4(x)::\phi^4(y):\big|_{x\to y} = \frac{1}{\epsilon^8}1 + \frac{1}{\epsilon^2}:\phi^2(y): + \frac{1}{\epsilon^4}:(\partial\phi)^2(y): + \frac{1}{\epsilon^4}:\phi^4(y): + \frac{1}{\epsilon^2}:\phi^6(y): + \dots \tag{13}$$

Here, $\epsilon = |x-y|$ is a short-distance regulator. The first four terms are not integrable, but can again be canceled by adding

$$A_0 + A_1\phi^2 + A_3\phi^4 + A_2(\partial\phi)^2 \tag{14}$$

counterterms to the Lagrangian. In general,

$$:O(x)::O(y):|_{x\to y} = \sum_n C_n(\epsilon) : O_n(y):, \tag{15}$$

where the $C_n(\epsilon) \sim \frac{1}{\epsilon^\alpha}$ are ordinary functions, and the $: O_n(y) :$ are normal-ordered local operators. Here,

$$\alpha = 2\Delta - \Delta_n, \tag{16}$$

where $[O] = \Delta$ and $[O_n] = \Delta_n$. This expansion is known as the operator product expansion, or the OPE. The structure of the OPE is such that the counterterms are always local. The OPE will be discussed in more detail in the problem set.

8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 20

5: THE RENORMALIZATION GROUP

The renormalization program has been shown to be a success operationally:

1. For a renormalizable theory, all ultraviolet divergences can be absorbed into redefinitions of bare quantities, which are not observable. What is meant by the set of physical observables is unambiguous: it is well-defined in terms of a small number of physical couplings, masses and so forth.

2. The requirement of renormalizability is a strong constraint on the physical theories we should consider. Theories satisfying this constraint included quantum electrodynamics, Yang-Mills theories and the standard model.

Conceptually, we want to understand why it works. Many of the older generation of physicists did not accept renormalization when it was first introduced operationally. There are several difficulties with this:

1. The process is perturbative in nature: it has to be carried out order-by-order in perturbation theory. Also, when carrying out perturbation theory, with a Lagrangian of the form

$$\mathscr{L} = -\frac{1}{2}\left(\partial\phi\right)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 - \mathscr{L}_{ct}, \tag{1}$$

for example, with $\mathscr{L}_{ct} = -\frac{1}{2}A\left(\partial\phi\right)^2 - \frac{1}{2}Bm^2\phi^2 - \frac{1}{3!}C\phi^3$. The coefficients A, B and C are found to be divergent, yet they are considered to be small quantities in the perturbation expansion.

2. Non-renormalizable terms are generated by loop corrections, such as the anomalous magnetic moment term, $\bar{\psi}\gamma^{\mu\nu}\psi F_{\mu\nu}$. Physically, we want to understand why this is not a bare term.

3. Conceptually, it is troublesome to assume that a particular theory like quantum electrodynamics should be valid to arbitrarily high energies.

5.1: WILSON'S APPROACH TO FIELD THEORIES

Let us consider Wilson's basic starting points, which should be viewed as the postulates of the theory.

1. A quantum field theory should be considered as an effective field theory, valid only in a certain range of energies. In particular, it should be defined with an ultraviolet cut-off Λ_0. For example, in quantum electrodynamics, we should choose $\Lambda_0 \sim 1\text{TeV}$, and we should only expect the theory to be valid for $E \ll \Lambda_0$.

2. Physical phenomena at a certain energy scale E are most appropriately described by the degrees of freedom at that scale. For example, for a scalar field

$$\phi(x) \leftrightarrow \phi(k), \quad k \in (0, \infty), \tag{2}$$

the most relevant degrees of freedom should be $\phi(k \sim E)$.

At the moment, these notions have only been expressed philosophically. The key is to actually implement these ideas procedurally. We will use the example of a single real scalar field for illustration, working in Euclidean signature. The simplest cut-off procedure is to restrict

$$|k| < \Lambda_0, \tag{3}$$

though other cut-off procedures can be considered. The basics ingredients of Wilson's approach are as followed:

1. Start with a bare Lagrangian defined at a scale Λ_0 :

$$\begin{aligned}\mathscr{L}[\Lambda_0] &= \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m_0^2\phi^2 + \frac{g_4^{(0)}}{4!}\phi^4 + \frac{g_6^{(0)}}{6!}\phi^6 + \ldots \\ &\equiv \mathscr{L}_0 + \mathscr{L}_{int},\end{aligned}$$

with $\mathscr{L}_0 \equiv \frac{1}{2}(\partial\phi)^2$, $\mathscr{L}_{int} \equiv \sum_i g_i^{(0)} O_i$, $O_n \equiv \frac{1}{n!}\phi^n$. Here, $\mathscr{L}[\Lambda_0]$ is only valid below a cut-off scale Λ_0. The bare couplings $\left\{g_i^{(0)}\right\}$ encode the unknown physics above Λ_0. $\mathscr{L}[\Lambda_0]$ includes all the interactions consistent with Lorentz symmetry and an additional discrete symmetry under $\phi \leftrightarrow -\phi$. That is, $\{O_i\}$ is a complete set of local operators consistent with a $\mathbb{Z}_2$ symmetry. If

$$[O_i] = \Delta_i, \tag{4}$$

then

$$\left[g_i^{(0)}\right] = d - \Delta_i \equiv \delta_i, \tag{5}$$

and so we introduce dimensionless couplings by

$$\lambda_i^{(0)} \equiv g_i^{(0)} \Lambda_0^{-\delta_i}. \tag{6}$$

We assume that $\lambda_i^{(0)} \sim O(1)$ for all i, treating all couplings on an equal footing, including the non-renormalizable ones; those satisfying $\delta_i < 0$.

2. The physics below the cut-off scale Λ_0 is fully captured by the path integral,

$$Z[J] = \int_{k<\Lambda_0} \mathfrak{D}\phi(k)\, e^{-S[\Lambda_0] - \int d^dx\, J\phi}, \tag{7}$$

where $S[\Lambda_0] = \int d^dx\, \mathscr{L}[\Lambda_0]$. The integration is only over $\phi(k)$ with $k < \Lambda_0$.

3. If we are interested in physics at the same energy scale $E < \Lambda_0$, we can integrate out degrees of freedom, for example $\phi(k)$ for $\Lambda < k < \Lambda_0$, where Λ is a new cut-off scale, $\Lambda \sim E$. More explicitly, we write

$$\phi(k) = \phi_\Lambda(k) + \tilde{\phi}(k), \tag{8}$$

where $\phi_\Lambda(k)$ has support for $k \in (0, \Lambda)$, and $\tilde{\phi}(k)$ has support for (Λ, Λ_0). Then,

$$\int_{k<\Lambda_0} \mathfrak{D}\phi(k) = \int_{k<\Lambda} \mathfrak{D}\phi_\Lambda(k) \int_{\Lambda<k<\Lambda_0} \mathfrak{D}\tilde{\phi}(k), \tag{9}$$

and hence we have

$$\begin{aligned}Z[J] &= \int_{k<\Lambda} \mathfrak{D}\phi_\Lambda(k)\, e^{\int d^dx\, J\phi_\Lambda} \int_{\Lambda<k<\Lambda_0} \mathfrak{D}\tilde{\phi}(k)\, e^{-S\left[\phi_\Lambda+\tilde{\phi},\Lambda_0\right] - \int d^dx\, J\tilde{\phi}} \\ &= \int_{k<\Lambda} \mathfrak{D}\phi_\Lambda(k)\, e^{-S[\phi_\Lambda,\Lambda] - \int d^dx\, J\phi_\Lambda},\end{aligned}$$

where $S[\phi_\Lambda, \Lambda]$ is the effective action at the new cut-off scale Λ, for the degrees of freedom contained in ϕ_Λ. $S[\phi_\Lambda, \Lambda]$ encodes the effects of all the high energy degrees of freedom, $\phi(k)$ with $k < \Lambda_0$ on the degrees of freedom below the new scale Λ. It captures all of the essential physics at the scale $k \sim \Lambda$. $S[\phi_\Lambda, \Lambda]$ is, of course, very complicated. We can parameterize it as

$$S[\phi_\Lambda, \Lambda] = \sum_{n=0}^{\infty} \int^{\Lambda} \left(\prod_{i=1}^{n} d^dk_i\right) \delta^{(d)}(k_1 + k_2 + \ldots + k_n) \times F_n(k_1, \ldots, k_n)\phi_\Lambda(k_1)\ldots\phi_\Lambda(k_n). \tag{10}$$

Here, the δ-function is a consequence of translational symmetry. Expanding $F_n(k_1, \ldots, k_n)$ as a power series of k_i, we can rewrite

$$\mathscr{L}[\phi_\Lambda, \Lambda] = \frac{1}{2}Z(\partial\phi)^2 + \sum_i g_i O_i, \tag{11}$$

where O_i is a complete set of local operators of ϕ_n allowed by symmetries. Here,

$$Z(\left\{g_i^{(0)}\right\}, \Lambda_0; \Lambda) \tag{12}$$

is the field renormalization from integrating out the degrees of freedom $\tilde{\phi}$, and

$$\left\{g_i(\left\{g_i^{(0)}\right\}, \Lambda_0; \Lambda)\right\} \tag{13}$$

are the physical couplings at the scale Λ. For example,

$$F_2(k_1, k_2) = F_2(k; \Lambda) = c_o + c_1 k^2 + \ldots \tag{14}$$

where we can write F_2 a function of a single $k < \Lambda$ because of the $\delta-$function in (10). In coordinate space, this contribute to the effective action can be written as

$$c_0\phi^2 + c_1(\partial\phi)^2 + \ldots. \tag{15}$$

So, after changing scale, the Lagrangian

$$\mathscr{L}_0\left[\Lambda_0\right] = \frac{1}{2}(\partial\phi)^2 + \sum_i g_i^{(0)} O_i \tag{16}$$

is replaced by

$$\mathscr{L}\left[\Lambda\right] = \frac{1}{2} Z(\partial\phi)^2 + \sum_i g_i O_i. \tag{17}$$

At first sight, Wilson's approach seems very complicated: $S\left[\phi_\Lambda, \Lambda\right]$, which is parameterized by an infinite number of interactions

$$\left\{g_i(\left\{g_i^{(0)}\right\}, \Lambda_0; \Lambda)\right\}, \tag{18}$$

depends on an infinite number of bare couplings,

$$\left\{g_i^{(0)}\right\} \tag{19}$$

at the scale Λ_0. To this extent, there is no predictive power. But it turns out there is some additional structure in this approach which makes it precisely the right way of thinking about field theories. First, a simple dimensional analysis can be used to give us some indication of this. We return to the bare action,

$$\begin{aligned} S\left[\Lambda_0\right] &= \int d^d x \, \frac{1}{2}(\partial\phi)^2 + \sum_i g_i^{(0)} O_i \\ &= \sum_i S_i, \end{aligned}$$

where $g_i^{(0)} \equiv \lambda_i^{(0)} \Lambda_0^{-\delta_i}$, and we suppose that we are interested in a process at an energy scale E. Then, if we assume all dimensional quantities are controlled by E; that is,

$$k \sim E, \quad \phi \sim E^{\frac{d-2}{2}}, \quad O_i \sim E^{\Delta_i}, \tag{20}$$

then we have

$$\begin{aligned} S_i \sim \int d^d x \, g_i^{(0)} O_i &\sim E^{\Delta_i - d} g_i^{(0)} = E^{-\delta_i} g_i^{(0)} \\ &\sim \lambda_i^{(0)} \left(\frac{E}{\Lambda_0}\right)^{-\delta_i}, \end{aligned}$$

where we have assumed that $\lambda_i^{(0)} \sim O(1)$, and we have that $\delta_i \equiv d - \Delta_i$. Then, for $E \ll \Lambda_0$, there are three cases:

1. $\delta_i > 0$, or $\Delta_i > d$: irrelevant coupling, $S_i \ll 1$ and the effect becomes negligibile for $E \ll \Lambda_0$.

2. $\delta_i = 0$, or $\Delta_i = d$: marginal coupling, $S_i \sim O(1)$.

3. $\delta_i < 0$, or $\Delta_i < d$: relevant coupling, $S_i \gg 1$.

So, for $E \ll \Lambda_0$, the non-renormalizable (irrelevant) bare couplings become suppressed. In particular, in the limit $\frac{E}{\Lambda_0} \longrightarrow 0$, or equivalently, $\Lambda_0 \longrightarrow \infty$, their contributions vanish. Therefore, the low energy physics only depends on a small number of renormalizable couplings, the marginal and relevant couplings, satisfying $\delta_i \geq 0$, even though in the bare Lagrangian, we took all couplings to appear at comparable strength. More explicitly, if we write

$$\left\{g_i^{(0)}\right\} = \left\{\rho_a^{(0)}\right\} + \left\{\kappa_b^{(0)}\right\} \tag{21}$$

where $\left\{\rho_a^{(0)}\right\}$ are the renormalizable and marginal couplings, and $\left\{\kappa_b^{(0)}\right\}$ are the non-renormalizable couplings, then in the limit $\frac{\Lambda}{\Lambda_0} \longrightarrow 0$,

$$\begin{aligned} g_i &= g(\left\{g_i^{(0)}\right\}, \Lambda_0; \Lambda) \\ &\longrightarrow g(\left\{\rho_a^{(0)}\right\}, \Lambda_0; \Lambda). \end{aligned}$$

This crude dimensional analysis will be made precise in later lectures.

Remarks:

1. In the Wilsonian approach, renormalization comes from integrating out the short-distance degrees of freedom.

2. No ultraviolet divergences are ever encountered.

3. There are underlying physical reasons for distinguishing renormalizable and non-renormalizable couplings.

8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 21

5.1: RENORMALIZATION GROUP FLOW

Consider the bare action defined at a scale Λ_0 :

$$S[\Lambda_0] = \int^{\Lambda_0} d^d x \, \frac{1}{2} (\partial\phi)^2 + \sum_i g_i^{(0)} O_i, \tag{1}$$

where O_i is a complete set of local operators formed from ϕ. The theory is specified by the set $\left\{ g_i^{(0)} \right\}$. As explained in the previous lecture, we can change the cutoff scale to some $\Lambda < \Lambda_0$ by integrating out the degrees of freedom in the interval (Λ, Λ_0). This gives

$$S[\Lambda] = \int^{\Lambda} d^d x \, \frac{1}{2} (\partial\phi)^2 + \sum_i g_i(\Lambda) O_i, \tag{2}$$

after redefining ϕ to absorb the field renormalization factor Z. This theory is specified by the set $\{g_i(\Lambda)\}$. Similarly, at another scale $\Lambda' < \Lambda$, we obtain $S[\Lambda']$, described by $\{g_i(\Lambda')\}$. These three actions, S_{Λ_0}, S_Λ and $S_{\Lambda'}$, should all describe the same physics at an energy scale $E < \Lambda' < \Lambda < \Lambda_0$. The relations between them can be found by integrating out the degrees of freedom explicitly in the path integral, giving

$$\begin{aligned} g_i(\Lambda) &= g_i(g_i^{(0)}, \Lambda_0; \Lambda), \\ g_i(\Lambda') &= g_i(g_i^{(0)}, \Lambda_0; \Lambda') \\ &= g_i'(g_i(\Lambda), \Lambda; \Lambda'). \end{aligned}$$

This process describes the renormalization group transformations, or the renormalization group flow: transformations between couplings at different scales to ensure they describe the same low energy physics. If we consider, for

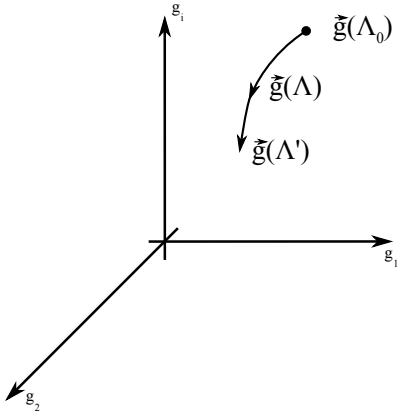


Figure 1: The renormalization flow as the flow in the space of all possible coupling parameterizations to ensure the same low-energy physics at different scales.

simplicity, the dimensionless couplings $\{\lambda_i(\Lambda)\}$ defined by $\lambda_i \equiv g_i \Lambda^{-\delta_i}$, differentiating gives

$$\Lambda \frac{d\lambda_i}{d\Lambda} = \beta_i(\{\lambda_j(\Lambda)\}) \tag{3}$$

where $\beta_i(\{\lambda_j(\Lambda)\}) = \frac{d}{d\ln\Lambda} \lambda_i(\{\lambda_j(\Lambda)\}, Z)\big|_{Z=1}$. It is important to note that the β_i are only functions of the dimensionless coupling constants $\{\lambda_j(\Lambda)\}$: they do not depend on Λ explicitly, as can be seen by considering

integrating out a fraction of the highest-energy modes in the path integral. The β-functions give the tangent vector of the flow, and depend only on the values of $\{\lambda_j\}$. Under a relabeling of couplings,

$$\tilde{\lambda}_i = \tilde{\lambda}_i(\{\lambda_j\}), \tag{4}$$

we have that

$$\tilde{\beta}_i(\{\tilde{\lambda}\}) = \sum_j \frac{d\tilde{\lambda}_i}{d\lambda_j}\beta_j(\{\lambda\}). \tag{5}$$

The $\beta-$functions can be computed explicitly from the path integral:

$$\begin{aligned} Z[J] &= \int_{|k|<\Lambda} \mathfrak{D}\phi\, e^{-S[\Lambda,\phi_\Lambda]-\int d^dx\, J\phi} \\ &= \int_{|k|<\Lambda'} \mathfrak{D}\phi_{\Lambda'}(k) \int_{\Lambda'<|k|<\Lambda} \mathfrak{D}\tilde{\phi}(k)\, e^{-S\left[\phi_{\Lambda'}+\tilde{\phi},\Lambda\right]-\int d^dx\, J(\phi_\Lambda+\tilde{\phi})} \\ &= \int_{|k|<\Lambda} \mathfrak{D}\phi_{\Lambda'}(k)\, e^{-S\left[\phi_{\Lambda'},\Lambda'\right]-\int d^dx\, J\phi_{\Lambda'}}. \end{aligned}$$

Now, if we let $\Lambda' \longrightarrow \Lambda - \delta\Lambda$, $S[\Lambda-\delta\Lambda] = S[\Lambda] + \delta S[\Lambda]$, we have

$$\Lambda\frac{dS_\Lambda}{d\Lambda} = F(S_\Lambda) \tag{6}$$

Expanding

$$S_\Lambda = \sum_i g_i O_i = \sum_i \lambda_i \Lambda^{\delta_i} O_i, \tag{7}$$

(6) gives us the β-functions for all couplings. As an example, let us consider the case of a free scalar field in four dimensions, with a cut-off at a scale Λ. Then, we have

$$S_\Lambda[\phi] = \int_{k<\Lambda} \frac{d^4k}{(2\pi)^4} f(k)\phi^*_\Lambda(k)\phi_\Lambda(k). \tag{8}$$

We expand $f(k)$ as a power series in k:

$$\begin{aligned} f(k) &= m_0^2 + k^2 + r_4k^4 + \dots \\ &= \lambda_m(\Lambda)\Lambda^2 + k^2 + \frac{\tilde{r}_4(\Lambda)}{\Lambda^2}k^4 + \dots, \end{aligned}$$

where the coefficient of k^2 can be chosen to one with a suitable normalization for ϕ_Λ. Here, $\lambda_m(\Lambda)$, $\tilde{r}_4(\Lambda), \dots$ are dimensionless couplings: $\left[\phi^2\right] = 2$, $\left[\left(\partial^2\phi\right)^2\right] = 6$, and so $\delta_m = 2$, $\delta_{r_4} = -2$, for example. We now let $\phi_\Lambda(k) = \phi_{\Lambda'}(k) + \tilde{\phi}(k)$ with $\tilde{\phi}(k)$ supported for $k \in (\Lambda', \Lambda)$ and $\phi_{\Lambda'}$ supported for $k \in (0, \Lambda')$. Then we have that

$$S_\Lambda[\phi_\Lambda] = S_\Lambda[\phi_{\Lambda'}] + S_\Lambda\left[\tilde{\phi}\right] + 2\int \frac{d^4k}{(2\pi)^4} f(k)\phi_{\Lambda'}(k)\tilde{\phi}(k), \tag{9}$$

where the last term is zero as $\phi_{\Lambda'}$ and ϕ_k have disjoint support. Integrating out $\tilde{\phi}$ only generates an overall constant for the path integral, and so

$$S_{\Lambda'}[\phi_{\Lambda'}] = S_\Lambda[\phi_{\Lambda'}] = \int_{k<\Lambda'} \frac{d^4k}{(2\pi)} f(k)\phi^*_{\Lambda'}(k)\phi_{\Lambda'}(k) \tag{10}$$

where $f(k)$ has not changed. That is,

$$\begin{aligned} f(k) &= m_0^2 + k^2 + r_4k^4 + \dots \\ &= \lambda_m(\Lambda')\Lambda'^2 + k^2 + \frac{\tilde{r}_4(\Lambda')}{\Lambda'^2}k^4 + \dots, \end{aligned}$$

and we conclude that

$$\begin{aligned}\lambda_m(\Lambda') &= \lambda_m(\Lambda)\left(\frac{\Lambda}{\Lambda'}\right)^2 = \lambda_m(\Lambda)\left(\frac{\Lambda}{\Lambda'}\right)^{-\delta_m} \quad \text{is a relevant operator,}\\ \tilde{r}_4(\Lambda') &= \tilde{r}_4(\Lambda)\left(\frac{\Lambda}{\Lambda'}\right)^{-2} = \tilde{r}_4(\Lambda)\left(\frac{\Lambda}{\Lambda'}\right)^{-\delta_m} \quad \text{is an irrelevant operator.}\end{aligned}$$

Similarly,

$$\begin{aligned}\beta_m &= \Lambda' \left.\frac{d\lambda_m(\Lambda')}{d\Lambda'}\right|_{\Lambda'\to\Lambda} = -2\lambda_m = -\delta_m\lambda_m < 0,\\ \beta_{r_4} &= \Lambda' \left.\frac{d\tilde{r}_4(\Lambda')}{d\Lambda'}\right|_{\Lambda'\to\Lambda} = 2\tilde{r}_4 = -\delta_m\tilde{r}_4 > 0.\end{aligned}$$

We note that dimensional quantities like m^2 and r_4 do not change at all in this instance, but that the dimensionless couplings flow as they are defined with respect to the cut-off scale. This does reflect the right physics: the relative importance of each term in $f(k)$ as we go to lower energies, or smaller k. That is,

$$\begin{aligned}\frac{m_0^2}{k^2} &\text{ becomes larger as } k \text{ becomes smaller,}\\ \frac{r_4 k^4}{k^2} &\text{ becomes smaller as } k \text{ becomes smaller.}\end{aligned}$$

We will now derive the full flow equation for $S_\Lambda[\phi]$. For this purpose, we write it as

$$S[\phi,\Lambda] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} G_\Lambda^{-1}(k)\phi(k)\phi(-k) + S_I[\phi,\Lambda] + U(\Lambda) \tag{11}$$

where $U(\Lambda)$ is a cosmological constant, and the propagator $G_\Lambda(k)$ satisfies

$$G_\Lambda(k) = \begin{cases}\frac{1}{k^2} & k \ll \Lambda,\\ 0 & k \gg \Lambda.\end{cases} \tag{12}$$

We have that

$$Z = \int \mathfrak{D}\phi(k)\, e^{-S_0[\phi,\Lambda]-\tilde{S}_I[\phi,\Lambda]}, \tag{13}$$

where $\tilde{S}_I = S_I + U$. There is now no need to impose an explicit cut-off when integrating over $\phi(k)$. It is clearly

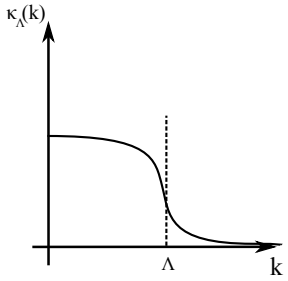


Figure 2: The propagator $G_\Lambda(k) = \frac{1}{k^2}\kappa_\Lambda(k)$ has a cut-off around $k \sim \Lambda$.

very complicated to obtain the flow equation for $\tilde{S}_I[\phi,\Lambda]$ by evaluating the path integral directly. We will instead require

$$\Lambda\frac{dZ[\Lambda]}{d\Lambda} = 0, \tag{14}$$

which is an equivalent statement. From this, we have

$$\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left\langle \phi(-k)\phi(k)e^{-\tilde{S}_I}\right\rangle \Lambda\frac{dG_\Lambda^{-1}}{d\Lambda} = \left\langle \Lambda\partial_\Lambda e^{-\tilde{S}_I}\right\rangle. \tag{15}$$

Here, $\langle \ldots \rangle = \frac{1}{Z_0} \int \mathfrak{D}\phi \ldots e^{-S_0}$, with $Z_0 = \int \mathfrak{D}\phi\, e^{-S_0}$. We would like to express the left-hand side of (15) more directly in terms of S_I. For this purpose, consider

$$0 = \int \mathfrak{D}\phi \, \frac{\delta}{\delta\phi(k)} \left(\phi(k) e^{-S_0 - \tilde{S}_I} \right). \tag{16}$$

From this, we have that

$$(2\pi)^4 \delta^{(4)}(0) \left\langle e^{-\tilde{S}_I} \right\rangle - G_\Lambda^{-1} \left\langle \phi(k)\phi(-k) e^{-\tilde{S}_I} \right\rangle + \left\langle \phi(k) \frac{\delta}{\delta\phi(k)} e^{-\tilde{S}_I} \right\rangle = 0. \tag{17}$$

The last term in this equation is still complicated. Consider further

$$0 = \int \mathfrak{D}\phi \, \frac{\delta^2}{\delta\phi(k)\delta\phi(-k)} \left(e^{-S_0 - \tilde{S}_I} \right). \tag{18}$$

From this, we have

$$(2\pi)^4 \delta^{(4)}(0) G_\Lambda^{-1} \left\langle e^{-\tilde{S}_I} \right\rangle - \left(G_\Lambda^{-1}\right)^2 \left\langle \phi(k)\phi(-k) e^{-\tilde{S}_I} \right\rangle - 2G_\Lambda^{-1} \left\langle \phi(k) \frac{\delta}{\delta\phi(k)} e^{-\tilde{S}_I} \right\rangle + \left\langle \frac{\delta^2}{\delta\phi(k)\delta\phi(-k)} e^{-\tilde{S}_I} \right\rangle = 0. \tag{19}$$

If we multiply 17 by $2G_\Lambda^{-1}$ and add the result to (19), we obtain

$$(2\pi)^4 \delta^{(4)}(0) G_\Lambda^{-1} \left\langle e^{-\tilde{S}_I} \right\rangle - \left(G_\Lambda^{-1}\right)^2 \left\langle \phi(k)\phi(-k) e^{-\tilde{S}_I} \right\rangle + \left\langle \frac{\delta^2}{\delta\phi(k)\delta\phi(-k)} e^{-\tilde{S}_I} \right\rangle = 0. \tag{20}$$

Eliminating $\left\langle \phi(k)\phi(-k) e^{-\tilde{S}_I} \right\rangle$ between (15) and (20) gives

$$\begin{aligned} \left\langle \Lambda \frac{d}{d\Lambda} e^{-S_I - U} \right\rangle &= -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \Lambda \frac{dG_\Lambda}{d\Lambda} \left\langle \frac{\delta^2}{\delta\phi(k)\delta\phi(-k)} e^{-S_I - U} \right\rangle \\ &\quad -\frac{1}{2} (2\pi)^4 \delta^{(4)}(0) \int \frac{d^4k}{(2\pi)^4} \Lambda \frac{d \log G_\Lambda}{d\Lambda} \left\langle e^{-S_I - U} \right\rangle. \end{aligned}$$

Here, the second term is a constant, and so we have

$$\Lambda \frac{d}{d\Lambda} U = \frac{1}{2} V_4 \Lambda \frac{d}{d\Lambda} \int \frac{d^4k}{(2\pi)^4} \log G_\Lambda(k), \tag{21}$$

where $V_4 = (2\pi)^4 \delta^{(4)}(0)$, and so

$$U(\Lambda) = U_0 + \frac{1}{2} V_4 \int \frac{d^4k}{(2\pi)^4} \log G_\Lambda(k) \tag{22}$$

where U_0 is independent of Λ, and

$$\Lambda \frac{d}{d\Lambda} e^{-S_I} = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \Lambda \frac{dG_\Lambda(k)}{d\Lambda} \frac{\delta^2}{\delta\phi(k)\delta\phi(-k)} e^{-S_I}, \tag{23}$$

or, equivalently,

$$\Lambda \frac{d}{d\Lambda} S_I = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \Lambda \frac{dG_\Lambda(k)}{d\Lambda} \left[\frac{\delta S_I}{\delta\phi(k)} \frac{\delta S_I}{\delta\phi(-k)} - \frac{\delta^2 S_I}{\delta\phi(k)\delta\phi(-k)} \right]. \tag{24}$$

8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 22

Firstly, we will summarize our previous results. We start with a bare Lagrangian,

$$\mathscr{L}\left[\Lambda_0,\phi\right]=\sum_i g_i^{(0)}O_i. \tag{1}$$

The path integral

$$Z=\int_{k<\Lambda_0}\mathfrak{D}\phi\, e^{-\int d^dx\mathscr{L}[\Lambda_0]} \tag{2}$$

describes all the physics below the cutoff Λ_0. Most often we are interested in physics at some energy scale $E\ll\Lambda_0$. $\mathscr{L}\left[\Lambda_0\right]$ is not convenient to use, as it contains the degrees of freedom $\phi(k):\ E<k<\Lambda_0$ which are not directly related to the physics at the scale E. However, we cannot simply discard them, as they have indirect effects, which can be taken into account by integrating them out. We write $\phi(k)=\phi_\Lambda(k<\Lambda)+\tilde{\phi}(\Lambda<k<\Lambda_0)$, and

$$Z_0=\int_{k<\Lambda}\mathfrak{D}\phi_\Lambda(k)\int_{\Lambda<k<\Lambda_0}\mathfrak{D}\tilde{\phi}(k)\, e^{-S\left[\phi_\Lambda+\tilde{\phi},\Lambda_0\right]}=\int_{k<\Lambda}\mathfrak{D}\phi_\Lambda(k)\, e^{-S[\phi_\Lambda,\Lambda]}, \tag{3}$$

where $S\left[\phi_\Lambda,\Lambda\right]=\int d^dx\sum_i g_i(\Lambda)O_i,\ g_i=g_i\left(\left\{g_i^{(0)}\right\},\Lambda_0;\Lambda\right)$. By varying Λ, we obtain a continuous family of $S\left[\phi_\Lambda,\Lambda\right]$ or $\{g_i(\Lambda)\}$. This is the renormalization group flow.

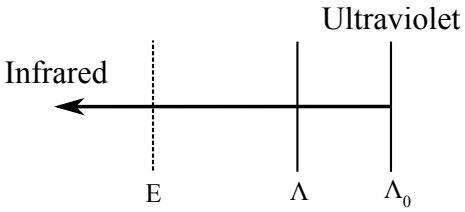


Figure 1: The renormalization flow from the cutoff Λ_0 in the ultraviolet to a cutoff Λ to study processes at an energy scale E in the infrared region.

Infinitesimally, we have

$$\Lambda\frac{dS_\Lambda}{d\Lambda}=F(S_\Lambda),\quad \Lambda\frac{d\lambda_i}{d\Lambda}=\beta_i\left(\{\lambda_j(\Lambda)\}\right), \tag{4}$$

where $\lambda_i=g_i\Lambda^{-\delta_i}$. The process means that all S_Λ should describe the same low-energy physics. By dimensional analysis, we expect that for $\frac{\Lambda}{\Lambda_0}\ll 1$,

$$\lambda_i(\Lambda)\sim\lambda_i(\Lambda_0)\left(\frac{\Lambda}{\Lambda_0}\right)^{-(d-\Delta_i)}, \tag{5}$$

and so we have three cases:

$$\lambda_i:\begin{cases}\Delta_i<d & \text{relevant},\\ \Delta_i=d & \text{marginal},\\ \Delta_i>d & \text{irrelevant}.\end{cases} \tag{6}$$

We expect the initial values of irrelevant couplings should not be important for $\frac{\Lambda}{\Lambda_0}\longrightarrow 0$. This rough argument can be substantiated by analyzing the flow equation in detail. It turns out that the flow equation can be written in a closed form, as we showed in the last lecture:

$$S\left[\phi_\Lambda,\Lambda\right]=S_0\left[\phi_\Lambda,\Lambda\right]+S_{int}\left[\phi_\Lambda,\Lambda\right], \tag{7}$$

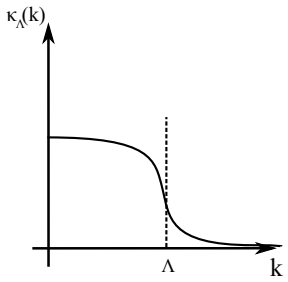


Figure 2: The propagator $G_\Lambda(k) = \frac{1}{k^2}\kappa_\Lambda(k)$ has a cutoff around $k \sim \Lambda$.

where $S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,\phi(k)\phi(-k)G_\Lambda^{-1}(k)$. If $G_\Lambda = \frac{1}{k^2}$, we have $S_0 = \frac{1}{2}\int d^4x\,(\partial\phi)^2$. We considered the case $G_\Lambda = \frac{1}{k^2}\kappa_\Lambda(k)$, where κ provides a cut-off at $k \sim \Lambda$. An example is a sharp cutoff $\kappa_\Lambda(k) = \Theta(1 - \frac{k}{\Lambda})$. This means $\phi(k)$ with $k > \Lambda$ do not propagate, and there is no need to impose $k < \Lambda$ explicitly in the path integral. Requiring the partition function to be independent of the choice of Λ led to the equation

$$\Lambda\frac{d}{d\Lambda}S_I = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,\Lambda\frac{dG_\Lambda(k)}{d\Lambda}\left[\frac{\delta S_I}{\delta\phi(k)}\frac{\delta S_I}{\delta\phi(-k)} - \frac{\delta^2 S_I}{\delta\phi(k)\delta\phi(-k)}\right]. \tag{8}$$

Remarks:

1. This equation is exact, and fully non-perturbative.

2. $\Lambda\frac{dG_\Lambda}{d\Lambda}$ is only supported near a thin shell of momentum around Λ. In fact, for $G_\Lambda = \Theta(1 - \frac{k}{\Lambda})$, we have $\Lambda\frac{dG_\Lambda}{d\Lambda} \propto \delta(k - \Lambda)$. Physically, this reflects that the flow equation is obtained by integrating out the degrees of freedom around Λ.

3. Expanding $S_I[\phi, \Lambda]$ in momentum space as

$$S_I[\phi,\Lambda] = \sum_{n=2}^{\infty}\frac{1}{n!}\int\left(\prod_{i=1}^{n}\frac{d^dk_i}{(2\pi)^4}\right)(2\pi)^4\delta^{(d)}(k_1 + k_2 + \ldots + k_n) \times g(k_1,\ldots,k_n;\Lambda)\phi(k_1)\ldots\phi(k_n). \tag{9}$$

(8) requires that

$$\Lambda\frac{d}{d\Lambda}g(k_1,\ldots k_n;\Lambda) = \sum_{\{I_1,I_2\}} g(-p,I_1;\Lambda)\Lambda\frac{dG_\Lambda(p)}{d\Lambda}g(p,I_2;\Lambda) - \frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\Lambda\frac{dG_\Lambda}{d\Lambda}g(q,-q,k_1,\ldots,k_n;\Lambda) \tag{10}$$

where $p = \sum_{k_i\in I_1} k_i$, and I_1, I_2 are disjoint subsets of momenta such that $I_1 \cup I_2 = \{k_1,\ldots,k_n\}$. $\sum_{\{I_1,I_2\}}$ is a sum over all possible ways to separate $\{k_1,\ldots,k_n\}$ into groups. Diagramatically, this is shown in figure 3.

This corresponds to integrating out a tree-level diagram and a one-loop momentum diagram respectively.

3. We can expand $S_I[\phi,\Lambda]$ in coordinate space:

$$S_I[\phi,\Lambda] = \int d^4x\sum_i g_i(\Lambda)O^i(x), \tag{11}$$

where the O_i form a complete set of local operators. From (8), we obtain

$$\Lambda\frac{dg_i}{d\Lambda} = \tilde{\beta}_i^j g_j + \tilde{\beta}_i^{jk}g_jg_k, \tag{12}$$

with $\tilde{\beta}_i^i = 0$. Using dimensionless couplings, $\lambda_i(\Lambda) = g_i(\Lambda)\Lambda^{-(d-\Delta_i)}$, we find

$$\beta_i \equiv \Lambda\frac{d\lambda_i}{d\Lambda} = \beta_i^j\lambda_j + \beta_i^{jk}\lambda_j\lambda_k, \tag{13}$$

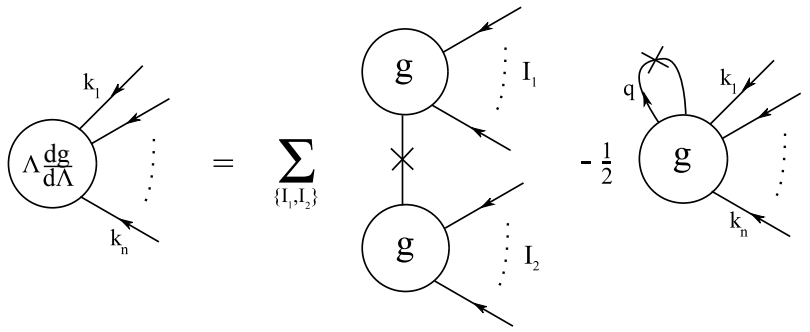


Figure 3: Diagramatic representation of (10), where the crossed vertex represents $\Lambda \frac{dG_\Lambda}{d\Lambda}(p)$.

where $\beta_i^j = (\Delta_i - d)\,\delta_i^j + \tilde{\beta}_i^j$, and the $\tilde{\beta}_i^j$ has no diagonal term. We see how this corresponds to the cases

$$\begin{cases} \Delta_i > d & \text{damping,} \\ \Delta_i = 0 & \text{marginal,} \\ \Delta_i < d & \text{growth.} \end{cases} \tag{14}$$

4. The flow equation, (8), and thus the resulting β-functions, are not unique. It can be written in many other, equivalent forms by using field redefinitions:

$$\phi(x) \longrightarrow \phi(x) + a\phi^2(x) + b\phi^3(x) + c\,(\partial\phi)^2 + \ldots. \tag{15}$$

Such field redefinitions lead to redefinitions of the couplings, but they should not change the physical observables. There is also a scheme dependence on the choice of cutoff functions, $\kappa_\Lambda(k)$. The equations (12) and (13) are an infinite number of coupled first-order differential equations. It is quite complicated to analyze them, and often requires the development of approximation methods.

Let us consider some general features of the flow:

1. Separate the couplings, as before, as $\{\lambda_i\} = \{\rho_a\} + \{\kappa_\alpha\}$, where $\{\rho_a\}$ are the relevant and marginal couplings, and $\{\kappa_\alpha\}$ are the irrelevant couplings. Then, for a generic theory with all $\lambda_i^{(0)} \sim O(1)$ at Λ_0, we have for $\frac{\Lambda}{\Lambda_0} \ll 1$, assuming the $\{\lambda_i\}$ do not become too large, that the $\{\kappa_\alpha\}$ only depend on the $\{\rho_a\}$. For example, if we consider two couplings, λ_4 and λ_6, for terms of the form ϕ^4 and ϕ^6 respectively, we have

$$\begin{aligned} \Lambda\frac{d\lambda_4}{d\Lambda} &= \lambda_6 + \ldots, \\ \Lambda\frac{d\lambda_6}{d\Lambda} &= 2\lambda_6 - \lambda_4^2 + \ldots \end{aligned}$$

The first term in the flow equation for λ_6 provides a damping when going into the infrared regime.

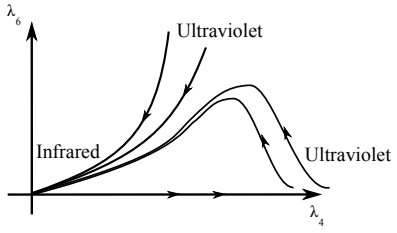


Figure 4: Damping and renormalization flow into the point $\lambda_4 = \lambda_6 = 0$ for the irrelevant coupling λ_6 and the marginally irrelevant coupling λ_4.

as $\frac{\Lambda}{\Lambda_0} \to 0$, $\lambda_6 \to \lambda_6 = \lambda_6(\lambda_4) \approx \frac{\lambda_4^2}{2}$. The flow of λ_4 is given by $\beta_4 = \Lambda\frac{d\lambda_4}{d\Lambda} = \frac{\lambda_4^2}{2} + O(\lambda_4^3)$, and so λ_4 is marginally irrelevant.

We can now make the connection to the standard renormalization procedure. We consider the ϕ^4-Lagrangian,

$$\mathscr{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2_{phys}\phi^2 - \frac{1}{4}\lambda_4^{(phys)}\phi^4 + \mathscr{L}_{ct}, \tag{16}$$

where m^2 and λ_4 are renormalized physical quantities which can be measure experimentally. They should be interpreted as being defined at a specific infrared scale. For example,

$$\lambda_4^{(phys)} = \lambda_4(\Lambda = 0). \tag{17}$$

We now keep $\lambda_4^{(phys)}$ and m^2_{phys} fixed and take the limit of the cutoff $\Lambda_0 \longrightarrow \infty$. All physical observables only depend on the renormalized quantities. In particular, $\lambda_4^{(0)}(\Lambda_0) \neq 0$, $\lambda_6^{(0)} = 0$. That is, we start along the horizontal axis. The Wilsonian approach tells us that such an initial condition is not important. Also note that λ_6 is not zero in the infrared, it is just determined by λ_4. λ_6 is related to six-particle scatterings, which of course have non-zero amplitude.

8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 23

5.2.2: Fixed Points of Renormalization Flow

What is the end point of a renormalization group flow as we take $\Lambda \longrightarrow 0$? This is one of the most important physical questions. Let us consider the possibilities:

1. Some couplings become very strong. This leads to the formation of bound states, and to new infrared degrees of freedom, as in quantum chromodynamics.

2. $\{\lambda_j\}$ has a fixed point. That is, $\beta_i(\{\lambda_j\}) = 0$, and so

$$\Lambda \frac{d\lambda_i}{d\Lambda} = 0, \tag{1}$$

meaning $\lambda_i(\Lambda) = \lambda_i^*$ is a constant, independent of scale. At a fixed point, a theory is scale invariant.

As a trivial example of this second case, we will consider a Gaussian fixed point. Let the Lagrangian be given by

$$\mathscr{L} = \frac{1}{2} (\partial\phi)^2, \tag{2}$$

describing a free, massless particle. In this case, $S_i = 0$, and all of the couplings satisfy $\lambda_i = 0$, and this is clearly invariant under the flow. In fact, our earlier discussion of RG flow and the notion of relevant and irrelevant couplings are defined implicitly with respect to this fixed point.

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{int}, \tag{3}$$

where $\mathscr{L}_0 = \frac{1}{2} (\partial\phi)^2$ and $\mathscr{L}_{int} = \sum_i g_i O_i$. The dimensions Δ_i of O_i are found from the canonical dimension of ϕ, which is defined using $\mathscr{L}_0$. For a generic fixed point, $\lambda_i(\Lambda) = \lambda_i^*$ constant, consider a small deviation from the fixed

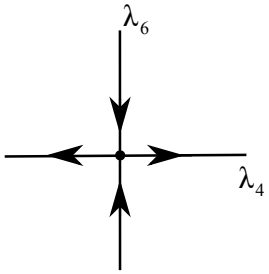


Figure 1: Flow diagram for the dimensionless couplings, λ_4 and λ_6. λ_4 is marginally irrelevant.

point,

$$\delta\lambda_i = \lambda_i - \lambda_i^*. \tag{4}$$

Then,

$$\begin{aligned} \Lambda \frac{d\delta\lambda_i}{d\Lambda} &= \beta_i \left(\left\{ \lambda_j^* + \delta\lambda_j \right\} \right) \\ &= \sum_j \left. \frac{d\beta_i}{d\lambda_j} \right|_{\lambda_j^*} \delta\lambda_j + \dots \\ &= \sum_j T_{ij} \delta\lambda_j + \dots \end{aligned}$$

with $T_{ij} = \frac{d\beta_i}{d\lambda_j}\Big|_{\lambda_j^*}$. We now consider the left eigenvectors, $e_i^{(\alpha)}$, of T, with eigenvalues t_α, where α indexes the eigenvectors. That is,

$$\sum_i e_i^{(\alpha)} T_{ij} = t_\alpha e_j^{(\alpha)}. \tag{5}$$

Now, we let

$$U_\alpha = \sum_i e_i^{(\alpha)} \delta\lambda_i. \tag{6}$$

Then

$$\Lambda \frac{dU_\alpha}{d\Lambda} = t_\alpha U_\alpha + O(U^2). \tag{7}$$

Disregarding the non-linear terms, we suppose at some Λ_0 that $U_\alpha = U_\alpha^{(0)}$. Then we have that

$$U_\alpha(\Lambda) = \left(\frac{\Lambda}{\Lambda_0}\right)^{t_\alpha} U_\alpha^{(0)}. \tag{8}$$

U_α is the scaling variable, and $-t_\alpha = [U_\alpha]$ is the scaling dimension, as defined with respect to the fixed point $\{\lambda^*\}$. We have that

$$-t_\alpha = [U_\alpha] = \begin{cases} > 0 & \text{relevant,} \\ = 0 & \text{marginal,} \\ < 0 & \text{irrelevant.} \end{cases} \tag{9}$$

Now,

$$\mathscr{L} = \mathscr{L}_* + \mathscr{L}_{int}, \tag{10}$$

where

$$\begin{aligned} \mathscr{L}_* &= \sum_i \lambda_i^* O_i, \\ \mathscr{L}_{int} &= \sum_i \delta\lambda_i O_i = \sum_\alpha U_\alpha \tilde{O}_\alpha, \end{aligned}$$

where $\tilde{O}_\alpha$ are the scaling operators with respect to $\mathscr{L}_*$. We have that $\left[\tilde{O}_\alpha\right] = d + t_\alpha$. Now consider the case of U_1, U_2 with $t_1 < 0$ (relevant) and $t_2 > 0$ (irrelevant). Then

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = M \begin{pmatrix} \delta\lambda_1 \\ \delta\lambda_2 \end{pmatrix}. \tag{11}$$

The **critical surface** is the submanifold in the space of couplings spanned by irrelevant couplings near a fixed

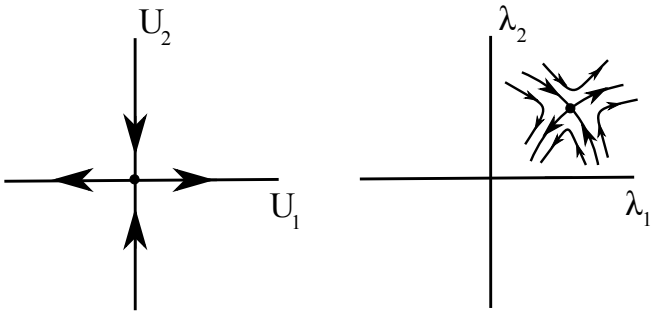


Figure 2: Flow in the U_1, U_2 plane and the λ_1, λ_2 plane about the fixed point λ^*.

point. Any point on the critical surface will flow to the fixed point. A point lying off the critical surface will flow away from it, as shown in figure 3. The critical surface has a finite co-dimension, since the number of relevant and marginal couplings is always finite.

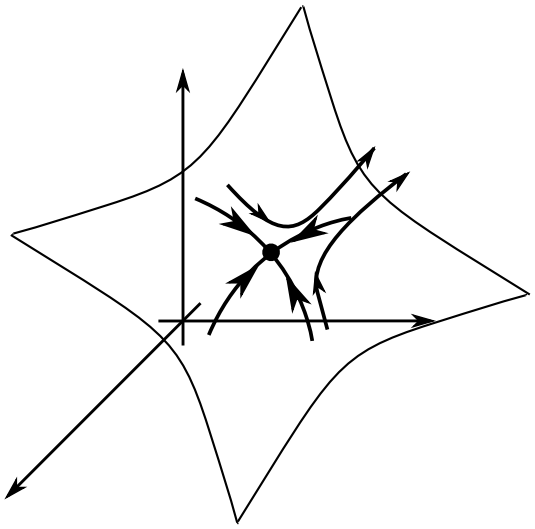


Figure 3: The critical surface submanifold. Couplings on the manifold are irrelevant and flow towards the fixed point. Couplings off the manifold flow away from it.

Let us consider, at the ultraviolet scale, Λ_0, $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{int}$, with $\mathscr{L}_0 = \frac{1}{2}(\partial\phi)^2$, and $\mathscr{L}_{int} = \sum_i \lambda_i O_i$ giving deviations from the fixed point.

$$O_i = \phi^2, (\partial\phi)^2, \phi^4, \ldots \tag{12}$$

are scaling operators with respect to $\mathscr{L}_0$. In general near $\{\lambda_i^*\}$, the spectrum of scaling operators $\{O_\alpha\}$ and their conformal dimensions will be very different from those at the Gaussian fixed point. Such non-trivial fixed points are very rare at a finite distance away from the Gaussian fixed point, particularly in $d \geq 3$. For this reason, they provide a very limited theoretical tool. We consider one famous example:

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{4!}\phi^4 \tag{13}$$

at Λ_0 in $d = 4$. Then we have

$$\beta_\lambda = c\lambda^2 + \ldots, \tag{14}$$

where c is a positive constant, giving a marginally irrelevant flow back to the Gaussian fixed point. If we now consider $d = 4 - \epsilon$, then

$$[\phi] = \frac{d-2}{2} = 1 - \frac{\epsilon}{2}, \quad [\lambda] = \epsilon, \tag{15}$$

for $\epsilon \ll 1$. Then we have

$$\beta_\lambda = -\epsilon\lambda + c\lambda^2 + \ldots, \tag{16}$$

and so, the fixed point is given by

$$\lambda_* = \frac{\epsilon}{c} \ll 1. \tag{17}$$

This fixed point is infinitesimally close to the Gaussian fixed point and can be studied using perturbation theory. We now conclude our discussion of the Wilsonian approach to renormalization. Our discussion has been more focused on the conceptual picture, rather than the explicity calculation. The Wilsonian approach is conceptually very appealing, but technically very awkward, as the couplings depend on the scale. We want to find how to see it from the conventional approach.

5.3: BETA-FUNCTIONS FROM THE TRADITIONAL APPROACH

We recall the procedure for renormalization, for example in ϕ^3-theory.

1. Firstly, we have the Lagrangian

$$\mathscr{L}(\phi_B, g_B, m_B) = \mathscr{L}(\phi, g, m) + \mathscr{L}_{ct}, \tag{18}$$

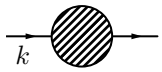
where

$$\mathscr{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{g}{6}\mu^{\frac{\epsilon}{2}}\phi^3 \tag{19}$$

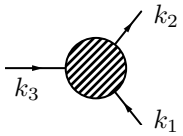
and

$$\mathscr{L}_{ct} = -\frac{1}{2}A(\partial\phi)^2 - \frac{1}{2}Bm^2\phi^2 + \frac{g}{6}\mu^{\frac{\epsilon}{2}}C\phi^3. \tag{20}$$

2. We impose the following renormalization conditions to determine the counter terms:

$\Pi(k^2) =$  (21)

$$\begin{aligned}\Pi(k^2 &= -m^2) = 0 \quad \text{(Physical mass condition)}\\ \Pi'(k^2 &= -m^2) = 0 \quad \text{(Physical field condition)}\end{aligned}$$

$V(k_1,k_2,k_3) =$  (22)

$$V(0,0,0) = g \quad \text{(Physical coupling condition)} \tag{23}$$

3. All physical observables can be expressed in terms of m^2 and g.

We consider more explicitly the one-loop expression for $\Pi(k^2)$ and $V(k_1,k_2,k_3)$:

$$\Pi(k^2) = -\frac{\alpha}{2}\left[\left(\frac{2}{\epsilon}+1\right)\left(\frac{k^2}{6}+m^2\right) + \int_0^1 dx\, D\log\left(\frac{4\pi\mu^2}{e^\gamma D}\right)\right] - Ak^2 - Bm^2, \tag{24}$$

where $D \equiv x(1-x)k^2 + m^2$, $\alpha \equiv \frac{g^2}{(4\pi)^3}$, and

$$\frac{1}{g}V(k_1,k_2,k_3) = 1 + \frac{\alpha}{2}\left[\frac{2}{\epsilon} + K\right] + C, \tag{25}$$

where $K = K(m^2,k_1,k_2,k_3,\mu)$. The explicit form of this function is not important here. In order to cancel the divergences, we require

$$\begin{aligned} A &= -\frac{\alpha}{6\epsilon} + a,\\ B &= -\frac{\alpha}{\epsilon} + b,\\ C &= -\frac{\alpha}{\epsilon} + c. \end{aligned}$$

where a, b and c are finite constants determined by the renormalization conditions. Therefore, $\Pi(k^2)$ and $V(k_1,k_2,k_3)$ are independent of the auxillary scale μ and depend only on m^2 and g. But we can actually choose a, b and c arbitrarily, and physical observables will be finite. The previously stated renormalization conditions will generally not be satisfied, and so m^2, ϕ and g are no longer the physical mass, field and couplings. They can be interpreted as parameters of the Lagrangian. The conditions given by

$$\begin{aligned} \Pi(k^2 = -m_{phys}^2) &= 0,\\ \Pi'(k^2 = -m_{phys}^2) &= 0,\\ V(0,0,0) &= g_{phys}, \end{aligned}$$

and are now solved by general

$$\begin{aligned} m_{phys} &= m_{phys}(m^2,g),\\ \phi_{phys} &= \phi_{phys}(m^2,g,\phi),\\ g_{phys} &= g_{phys}(m^2,g), \end{aligned}$$

which are all finite relations. For example, if we set $0 = a = b = c$, we find

$$g_{phys} = g\left[1 + \frac{\alpha}{2}K(m^2, \mu; k_i = 0)\right], \tag{26}$$

where

$$K = \int_0^1 dx_1 dx_2 dx_3\, \delta(x_1 + x_2 + x_3 - 1) \log\left(\frac{4\pi\mu^2}{e^\gamma \tilde{D}}\right), \tag{27}$$

with

$$\tilde{D} \equiv m^2 + x_2x_3k_1^2 + x_1x_3k_2^2 + x_1x_2k_3^2, \tag{28}$$

and so

$$g_{phys} = g\left[1 + \frac{\alpha x}{2}\log\left(\frac{4\pi\mu^2}{m^2}\right)\right], \tag{29}$$

where x is a numerical constant. A similar result holds for m^2_{phys}. We note that, in general, the $\mu-$dependence does not disappear in such relations. In practice, we choose μ, and from m_{phys} and g_{phys}, that is, from the measured quantities, we determine g and m and hence all other observables. The physical observables, of course, like m_{phys} and g_{phys} do not depend on μ. So we have $g(\mu)$ and $m^2(\mu)$, and

$$\mathscr{L}(\phi_B, g_B, m_B) = \mathscr{L}(\phi, g, m) + \mathscr{L}_{ct}. \tag{30}$$

Here, the left-hand side is μ-independent. The splitting on the right-hand side requires introducing a scale μ. ϕ, g and m^2 should be $\mu-$dependent, so that the physical observables are $\mu-$independent, but the finite part of the split is arbitrary. A different choice of a,b and c is a different choice of scheme: the one we have been using until now is the on-shell scheme. Different schemes give different $g(\mu)$, $\phi(\mu)$ and $m^2(\mu)$. Again, the physics should not depend on such a choice, corresponding to the different ways of doing the splitting.

8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 24

Let us consider some of the renormalization schemes we discussed in the previous lecture. A particularly convenient renormalization scheme in dimensional regularization is minimal subtraction (MS). In this case, we take $a = b = c = 0$. This gives

$$\begin{aligned}\Pi_{MS}(k^2) &= -\frac{\alpha}{2}\left[\frac{k^2}{6} + m^2 + \int_0^1 dx\, D \log\left(\frac{4\pi\mu^2}{e^\gamma D}\right)\right], \\ \frac{1}{g}V_{MS}(k_1,k_2,k_3) &= 1 + \frac{\alpha}{2}\int dF_3 \log\left(\frac{4\pi\mu^2}{e^\gamma \tilde{D}}\right),\end{aligned}$$

where $\int dF_3 \equiv \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \delta(x_1+x_2+x_3-1)$, $D \equiv x(1-x)k^2 + m^2$ and $\tilde{D} \equiv m^2 + x_2x_3k_1^2 + x_1x_3k_2^2 + x_1x_2k_3^2$. The k-dependence of $g(\mu)$ and $m(\mu)$ should be such that physical observables are independent of μ. In the $\overline{\text{MS}}$ scheme, we take

$$\frac{1}{\epsilon} \longrightarrow \frac{1}{\epsilon} - \frac{1}{2}\log\left(\frac{4\pi^2}{e^\gamma}\right), \tag{1}$$

giving

$$\Pi_{\overline{MS}}(k^2) = -\frac{\alpha}{2}\left[\frac{k^2}{6} + m^2 + \int_0^1 dx\, D \log\left(\frac{\mu^2}{D}\right)\right]. \tag{2}$$

In the on-shell scheme, we have

$$\mathscr{L}(\phi_B, m_B, g_B) = -\frac{1}{2}Z_\phi^{(on)}\left(\partial\phi_{phys}\right)^2 - \frac{1}{2}Z_m^{(on)}m_{phys}^2\phi_{phys}^2 + \frac{g_{phys}}{6}\mu^{\frac{\epsilon}{2}}Z_g^{(on)}\phi_{phys}^3. \tag{3}$$

In the MS scheme, we have

$$\mathscr{L}(\phi_B, m_B, g_B) = -\frac{1}{2}Z_\phi^{(MS)}\left(\partial\phi_{MS}\right)^2 - \frac{1}{2}Z_m^{(MS)}m_{MS}^2\phi_{MS}^2 + \frac{g_{MS}}{6}\mu^{\frac{\epsilon}{2}}Z_g^{(MS)}\phi_{MS}^3. \tag{4}$$

Finally, in the $\overline{\text{MS}}$ scheme, we have

$$\mathscr{L}(\phi_B, m_B, g_B) = -\frac{1}{2}Z_\phi^{(\overline{MS})}\left(\partial\phi_{\overline{MS}}\right)^2 - \frac{1}{2}Z_m^{(\overline{MS})}m_{\overline{MS}}^2\phi_{\overline{MS}}^2 + \frac{g_{\overline{MS}}}{6}\mu^{\frac{\epsilon}{2}}Z_g^{(\overline{MS})}\phi_{\overline{MS}}^3. \tag{5}$$

For the fields, we have

$$\begin{aligned}\phi_B &= \left(Z_\phi^{(on)}\right)^{\frac{1}{2}}\phi_{phys} \\ &= \left(Z_\phi^{(MS)}\right)^{\frac{1}{2}}\phi_{MS} \\ &= \left(Z_\phi^{(\overline{MS})}\right)^{\frac{1}{2}}\phi_{\overline{MS}}.\end{aligned}$$

The three field renormalizations here are all divergent, but their ratios are finite. Why do we not just use the on-shell scheme?

1. There are instances where it can't be used, for example, if $m_{phys} = 0$.

2. Other schemes can be more convenient in certain settings.

3. More seriously, consider $\left|k^2\right| \gg m^2$. Then we have

$$D \approx x(1-x)k^2, \tag{6}$$

and

$$\log \frac{D}{D_0} \approx \log \frac{k^2}{m^2} + \dots. \tag{7}$$

Hence,

$$\Pi(k^2) \approx \frac{\alpha}{12} k^2 \log\left(\frac{k^2}{m^2}\right), \tag{8}$$

which can be large compared with k^2, and so perturbation theory is no longer a good approximation. Similarly,

$$\frac{1}{g} V(k_1, k_2, k_3) \approx 1 + \alpha \log\left(\frac{k^2}{m^2}\right), \tag{9}$$

and perturbation theory is not a good approximation for large k^2. Introducing μ allows us to address this problem: if we choose $\mu \sim k$, no such logarithm arises.

If we choose μ appropriately, that is, to be comparable to the momentum scale of the physical process, we can improve our perturbation expansion. As we will see shortly,

1. $g(\mu)$ and $m(\mu)$ can be considered as the counterparts of the scale-dependent coupling constants of the Wilsonian approach.

2. The reason we get large logarithmic terms in the on-shell scheme is that we are trying to use coupling defined at one scale to describe physics at very different scales. We will return to this point later with a physical explanation.

Let us consider the structure of general correlation functions. Having looked at $\Pi(k^2)$ and $V(k_1, k_2, k_3)$ at the one-loop level, let us now look at general connected Greens functions in some renormalization scheme, such as the MS scheme:

$$G_n(x_1, \dots, x_n) = \langle \Omega \left| T(\phi(x_1) \dots \phi(x_n)) \right| \Omega \rangle, \tag{10}$$

where $\phi(x)$ is the renormalized field. Then

$$G_n(\{x\}, g(\mu), m(\mu); \mu) = G_n(\{x\}, \lambda_i(\mu); \mu) \tag{11}$$

where $\lambda_i(\mu)$ are dimensionless coupling corresponding to g and m^2, defined with respect to μ. For example, $\lambda_m = \frac{m^2(\mu)}{\mu^2}$. We consider also

$$G_n^{(B)}(x_1, \dots, x_n) = \langle \Omega \left| T(\phi_B(x_1) \dots \phi_B(x_n)) \right| \Omega \rangle. \tag{12}$$

Then

$$G_n^{(B)}(\{x\}, g_B, m_B; \Lambda_0), \tag{13}$$

where Λ_0 is an ultraviolet cut-off, independent of μ. Since $\phi_B = Z_\phi^{\frac{1}{2}} \phi$, we have that

$$G_n^{(B)} = Z_\phi^{\frac{n}{2}} G_n, \tag{14}$$

and so

$$\mu \frac{d}{d\mu} (Z_\phi^{\frac{n}{2}} G_n) = 0. \tag{15}$$

Introducing $\gamma \equiv \frac{1}{2} \mu \frac{d}{d\mu} \log Z_\phi$, we have

$$\left(\mu \frac{d}{d\mu} + n\gamma\right) G_n = 0, \tag{16}$$

that is,

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_i \frac{\partial}{\partial \lambda_i} + n\gamma\right) G_n = 0 \tag{17}$$

where $\beta_i \equiv \mu \frac{d\lambda_i}{d\mu}$. This is the Callan-Symanzik equation. We note that $\gamma = \gamma(\{\lambda_i\})$ and $\beta_i = \beta_i(\{\lambda_i\})$. From the Callan-Symanzik equation, we can express $G_n(\{x\}, \lambda_i(\mu'); \mu')$ in terms of $G_n(\{x\}, \lambda_i(\mu); \mu)$. First, consider $\gamma = 0$. Then we have

$$\frac{d}{d\mu} G_n = 0, \tag{18}$$

and so

$$G_n(\{x\}, \lambda_i(\mu'); \mu') = G_n(\{x\}, \lambda_i(\mu); \mu). \tag{19}$$

These are the running coupling constants we had earlier. In the case $\gamma \neq 0$, γ does not depend on μ explicitly, but it does indirectly through the $\gamma = \gamma(\lambda_j(\mu))$. In this case, we have

$$G_n(\{x\}, \lambda_i(\mu'); \mu') = \exp\left[-n \int_{\log \mu}^{\log \mu'} d\xi \, \gamma(\lambda_j(\xi))\right] \times G_n(\{x\}, \lambda_i(\mu); \mu). \tag{20}$$

γ captures how the definition of ϕ changes as we change μ. In the case that there is only one coupling g, that is, if $m = 0$, then since $\mu \frac{dg}{d\mu} = \beta$,

$$d\xi = \frac{dg}{\beta}, \tag{21}$$

$$G_n(\{x\}, \lambda_i(\mu'); \mu') = \exp\left[-n \int_{\log \mu}^{\log \mu'} \frac{dg'}{\beta(g')} \gamma(g')\right] \times G_n(\{x\}, \lambda_i(\mu); \mu). \tag{22}$$

Let us consider some applications of this:

1. For the high momentum behviour, consider

$$G_2(p, \lambda(\mu'); \mu') = \eta^{-2}(\mu', \mu) G_2(p, \lambda_i(\mu); \mu) \tag{23}$$

where $\eta^{-n} \equiv \exp\left[-n \int_{\log \mu}^{\log \mu'} d\xi \, \gamma(\lambda_j(\xi))\right]$. In particular,

$$G_2(\kappa p, \lambda(\kappa \mu'); \kappa \mu') = \eta^{-2}(\kappa \mu, \mu) G_2(\kappa p, \lambda_i(\mu); \mu). \tag{24}$$

This gives a one-dimensional group,

$$G_2(p, \lambda_i(\mu); \mu) = \frac{1}{p^2} f_2\left(\frac{p}{\mu}, \lambda_i(\mu)\right) \tag{25}$$

and so

$$\begin{aligned} G_2(\kappa p, \lambda_i(\kappa \mu); \kappa \mu) &= \frac{1}{\kappa^2 p^2} f_2\left(\frac{p}{\mu}, \lambda_i(\kappa \mu)\right) \\ &= \frac{1}{\kappa^2} G_2(p, \lambda_i(\kappa \mu); \mu). \end{aligned}$$

Equivalently, we have

$$G_2(\kappa p, \lambda_i(\mu); \mu) = \frac{\eta^2(\kappa \mu, \mu)}{\kappa^2} G_2(p, \lambda_i(\kappa \mu); \mu). \tag{26}$$

2. At a fixed point, $\beta_i = 0$, and so λ_i is a constant, so $\gamma(\{\lambda_i\})$ is also a constant, and

$$\left(\mu \frac{d}{d\mu} + n\gamma\right) G_n(\{x\}) = 0. \tag{27}$$

Consider, for example, $G_2(x; \mu) = \mu^2 f(\mu x)$. Then we have that

$$\left(y \frac{d}{dy} + 2\Delta\right) f(y) = 0, \tag{28}$$

where $\Delta = 2 + \gamma$. Hence,

$$f(y) = \frac{c}{y^{2\Delta}} \tag{29}$$

and

$$G_2(x;\mu) = \frac{c'}{x^{2\Delta}}. \tag{30}$$

More generally, for

$$G_n\left(\{x\};\mu\right) = \mu^{n\Delta_0} f_n\left(\{\mu x\}\right) \tag{31}$$

where Δ_0 is the canonical dimension, from the Callan-Symanzik equation, we have that f_n should satisfy

$$f_n\left(\{\lambda y\}\right) = \lambda^{-n\Delta} f_n\left(\{y\}\right) \tag{32}$$

with $\Delta = \Delta_0 + \gamma$.

In summary, we have introduced the renormalization scale μ, and $\lambda_i(\mu)$ are scale-dependent couplings, given by the renormalization group flow. Different choices of μ correspond to different descriptions of physical observables. However, the physical observables themselves do not depend on the choice of μ.

8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 25

5.3.2: Computation of Beta-Functions

We consider the beta functions for the mass m and the coupling g:

$$\beta_g = \mu \frac{dg}{d\mu}, \quad \beta_m = \mu \frac{d\lambda_m}{d\mu}, \tag{1}$$

where $\lambda_m = \frac{m^2(\mu)}{\mu^2}$. Note that the coupling constants in different renormalization schemes are generally different. In general, we have

$$\begin{aligned} \{\lambda_i\}: &\quad \text{scheme 1}, \\ \{\tilde{\lambda}_i\}: &\quad \text{scheme 2}. \end{aligned}$$

In the problem set, we will see how the $\beta-$functions transform. In particular, the first two terms are universal.

Example 1: $g\phi^3$ in $d = 6$ with MS scheme

$$\begin{aligned} \mathscr{L} &= -\frac{1}{2}(\partial\phi_B)^2 - \frac{1}{2}m_B^2\phi_B^2 + \frac{g_B}{6}\phi_B^3 \\ &= -\frac{1}{2}(1+A)(\partial\phi)^2 - \frac{1}{2}m^2(1+B)\phi^2 + \frac{g}{6}\mu^{\frac{\epsilon}{2}}(1+C)\phi^3, \end{aligned}$$

with

$$\begin{aligned} \phi_B &= (1+A)^{\frac{1}{2}}\phi, \\ m_B &= (1+A)^{-\frac{1}{2}}(1+B)^{\frac{1}{2}} m, \\ g_B &= g\mu^{\frac{\epsilon}{2}}(1+A)^{\frac{3}{2}}(1+C), \end{aligned}$$

$A = -\frac{\alpha}{6\epsilon}$, $B = -\frac{\alpha}{\epsilon}$ and $C = -\frac{\alpha}{\epsilon}$, up to $O(\alpha^2)$. The key is to note that the bare quantities should be independent of μ :

$$\mu\frac{dm_B}{d\mu} = 0, \quad \mu\frac{dg_B}{d\mu} = 0. \tag{2}$$

This leads to results for β_g and β_m. In general, in the case of dimensional regularization and minimal subtraction,

$$g_i^{(B)} = \mu^{\delta_i(\epsilon)}\left[\lambda_i(\mu) + \sum_{n=1}^{\infty}\epsilon^{-n}G_i^{(n)}(\lambda_j)\right] \tag{3}$$

where $\delta_i(\epsilon) = \delta_i + a_i\epsilon$: the last correction is due to dimensional regularization. From (2), we have

$$\beta_i(\epsilon) = \mu\frac{d\lambda_i}{d\mu}. \tag{4}$$

We can expand

$$\beta_i(\epsilon) = \beta_i + \epsilon\alpha_i \tag{5}$$

where the first term is the $\beta-$function and the second term again comes from dimensional regularization. If we take the $\mu-$derivative of (3), we find

$$\begin{aligned} 0 &= \delta_i(\epsilon)\left[\lambda_i + \sum_{i=1}^{\infty}\epsilon^{-n}G_i^{(n)}\right] \\ &\quad + \left[\beta_i(\epsilon) + \sum_{i=1}^{\infty}\epsilon^{-n}\frac{\partial G_i^{(n)}}{\partial\lambda_j}\beta_j(\epsilon)\right]. \end{aligned}$$

Equating both sides of the above equation order by order in ϵ, we find

$$\begin{aligned}(\delta_i + a_i\epsilon)\left[\lambda_i + \epsilon^{-1}G_i^{(0)} + \epsilon^{-2}G_i^{(2)} + \ldots\right] \\ + \left[\beta_i + \epsilon\alpha_i + \epsilon^{-1}\frac{\partial G_i^{(0)}}{\partial\lambda_j}\left(\beta_j + \epsilon\alpha_j\right) + \epsilon^{-2}\ldots\right] &= 0,\end{aligned}$$

and so, at $O(\epsilon)$,

$$\alpha_i = -\lambda_i a_i, \tag{6}$$

(note that we are not invoking the summation convention here,) and, at $O(\epsilon^2)$,

$$\delta_i\lambda_i + a_i G_i^{(1)} + \beta_i + \sum_j \alpha_j \frac{\partial G_i^{(1)}}{\partial\lambda_j} = 0, \tag{7}$$

or, equivalently,

$$\beta_i = -\delta_i\lambda_i - a_i G_i^{(1)} + \sum_j \frac{\partial G_i^{(1)}}{\partial\lambda_j} a_j\lambda_j. \tag{8}$$

We note that the β_i are determined by simple-pole residuces of the counter-terms, and that at $O(\epsilon^{-n})$ for $n \geq 1$, the constraints determine $G_i^{(n)}$ for $n \geq 2$ in terms of $G_i^{(1)}$. We now return to our discussion of the $g\phi^3-$theory. Here,

$$\begin{aligned}g_1 &= \alpha_B = \alpha\mu^\epsilon\left(1+A\right)^{-3}\left(1+B\right), \\ g_2 &= m_B^2 = \mu^2\lambda_m\left(1 - \frac{5\alpha}{6\epsilon} + \ldots\right),\end{aligned}$$

where $\alpha \equiv \frac{g^2}{(4\pi)^3}$, and so we find

$$\delta_1(\epsilon) = \epsilon, \quad \delta_1 = 0, \quad a_1 = 1, \quad G_1^{(1)} = -\frac{3}{2}\alpha^2, \tag{9}$$

and

$$\delta_2(\epsilon) = 2, \quad \delta_2 = 2, \quad a_2 = 0, \quad G_1^{(1)} = -\frac{5}{6}\lambda_m\alpha. \tag{10}$$

From this, we find for the $\beta-$functions,

$$\begin{aligned}\beta_\alpha &= \frac{3}{2}\alpha^2 - 3\alpha^2 = -\frac{3}{2}\alpha^2, \\ \beta_m &= -2\lambda_m - \frac{5}{6}\lambda_m\alpha = -2\lambda_m - \frac{5}{6}\lambda_m\alpha + \ldots.\end{aligned}$$

Let us consider the physical implications of these equations.

1. At weak coupling, $\alpha \ll 1$, β_m is dominated by the first term, $\beta_m \approx -2\lambda_m$. This gives the dimension in the absence of the interaction, which implies the familiar behaviour

$$\lambda_m\left(\mu\right) = \lambda_m\left(\mu_0\right)\left(\frac{\mu_0}{\mu}\right)^2, \tag{11}$$

and so $\lambda_m(\mu)$ grows quadratically as we decrease μ.

2. α is marginal in the absence of interactions, and so, interactions are important to determine the leading contribution. For $g\phi^3$, $\beta_2 < 0$, and the coupling is marginally relevant: α becomes stronger going into the infrared, as we decrease μ, and stronger going into the ultraviolet, as we increase μ.

We now integrate

$$\mu\frac{d\alpha}{d\mu} = -\frac{3}{2}\alpha^2, \tag{12}$$

which is equivalent to

$$d\left(\frac{1}{\alpha}\right) = \frac{3}{2}d\log\mu. \tag{13}$$

Suppose that $\alpha(\mu_0) = \alpha_0$. Then we have

$$\frac{1}{\alpha(\mu)} - \frac{1}{\alpha_0} = \frac{3}{2}\log\frac{\mu}{\mu_o}, \tag{14}$$

and hence,

$$\alpha(\mu) = \frac{\alpha_0}{1 + \frac{3\alpha_0}{2}\log\frac{\mu}{\mu_0}}. \tag{15}$$

In particular, as $\mu \longrightarrow \infty$, $\alpha(\mu) \longrightarrow 0$. This is asymptotic freedom. $\alpha(\mu) \longrightarrow \infty$ when $\frac{3\alpha_0}{2}\log\frac{\mu}{\mu_0} = -1$. That

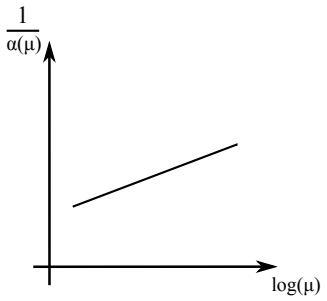


Figure 1: We find a linear relationship between $\alpha^{-1}(\mu)$ and $\log(\mu)$ with a positive gradient for the ϕ^3 theory.

is, when $\mu = \mu_0 e^{-\frac{2}{3\alpha_0}} \equiv \Lambda$. We note that this discussion only applies to $\alpha(\mu) \ll 1$. Of course, our one-loop approximation already breaks down before Λ is reached. Nevertheless, Λ provides a characteristic scale for the system. Λ is independent of μ_0. We can rewrite

$$\alpha(\mu) = \frac{2}{3}\frac{1}{\log\frac{\mu}{\Lambda}}, \tag{16}$$

and instead of specifying $\alpha(\mu_0) = \alpha_0$, we can simply specify Λ. The system does not have any dimensionless coupling. Rather, it has only a scale Λ. This is known as **dimensional transmutation**.

Now let us go back to the issue of the large logarithms encountered in the on-shell scheme when $k^2 \gg m^2$. We encountered $\alpha\log\frac{k^2}{m^2}$ at the one-loop level. However, it goes away if we choose $\mu \sim k$. We want to understand why this is, and why we can still trust perturbation theory. To see what is happening, let us consider, for $\mu \sim k$

$$\alpha(\mu_0 = m) = \alpha_0, \tag{17}$$

so that

$$\alpha(\mu) = \frac{\alpha}{1 + \frac{3}{2}\alpha_0\log\frac{\mu}{m}} \sim \alpha_0\sum_{n=0}^{\infty}\left(\alpha_0\log\frac{\mu}{m}\right)^n. \tag{18}$$

These are exactly the logarithmic terms we have seen before. They were just transferred to the relation between $\alpha(\mu \sim k)$ and α_0. The higher loop correction give higher powers in $\alpha_0\log\frac{\mu}{m}$. As a perturbation series in α_0, the last expression becomes bad when $\alpha_0\log\frac{\mu}{m}$ becomes large, but through the miracle of the renormalization group flow, by integrating the $\beta-$function, we have essentially resummed this bad series as far as $\alpha(\mu)$ remains small for all μ. This remarkable result is the essence of the renormalization group flow, which clearly also applies to the Wilsonian approach.

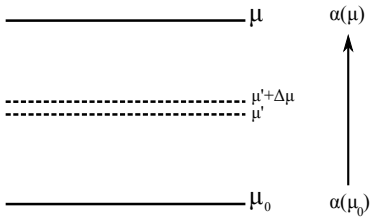


Figure 2: $\alpha(\mu)$ and $\alpha(\mu_0)$ are separated by large logarithms, but if we take infinitesimal steps, $\Delta\alpha = \alpha^2(\mu)\log\frac{\mu+\Delta\mu}{\mu} \sim \alpha^2(\mu)\frac{\Delta\mu}{\mu}$, and so for $\alpha^2(\mu)$ small, we can ignore the higher order terms.

8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 26

5.3.3: Beta-Functions of Quantum Electrodynamics

In the case of quantum electrodynamics, we have the Lagrangian

$$\mathscr{L} = -\frac{1}{4}F^B_{\mu\nu}F_B^{\mu\nu} - i\bar{\psi}_B\left(\gamma^\mu\left(\partial_\mu - ie_B A^B_\mu\right) - m_B\right)\psi_B, \tag{1}$$

with $\psi_B = Z_2^{\frac{1}{2}}\psi$, $A^B_\mu = Z_3^{\frac{1}{2}}A_\mu$, $m_B = m + \delta m$, and $e_B = Z_3^{-\frac{1}{2}}e\mu^{\frac{\epsilon}{2}}$ or $\alpha_B = Z_3^{-1}\alpha\mu^\epsilon$, where $\alpha = \frac{e^2}{4\pi}$. From our earlier result that $Z_3 = 1 - \frac{2\alpha}{3\pi}\frac{1}{\epsilon}$, we have

$$\alpha_B = \mu^\epsilon\left[\alpha + \frac{2\alpha^2}{3\pi}\frac{1}{\epsilon} + \ldots\right], \tag{2}$$

and

$$\beta_\alpha = -\frac{2\alpha^2}{3\pi} + \frac{4\alpha^2}{3\pi} = \frac{2\alpha^2}{3\pi} + O(\alpha^3). \tag{3}$$

Hence, the running of $\alpha(\mu)$ is given by

$$\frac{1}{\alpha(\mu)} = \frac{1}{\alpha(\mu_0)} - \frac{2}{3\pi}\log\frac{\mu}{\mu_0}, \tag{4}$$

or, equivalently,

$$\alpha(\mu) = \frac{\alpha_0}{1 - \frac{2\alpha_0}{3\pi}\log\frac{\mu}{\mu_0}}. \tag{5}$$

In quantum electrodynamics, $\alpha(\mu)$ increases as μ is increased, and $\alpha(\mu)$ decreases as μ decreases. In particular,

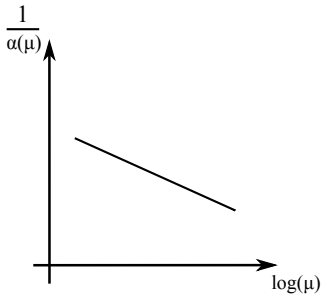
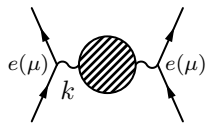


Figure 1: We find a linear relationship between $\alpha^{-1}(\mu)$ and $\log(\mu)$ with a negative gradient for quantum electrodynamics.

the Landau pole at $\alpha(\mu) \longrightarrow \infty$ is given by

$$\mu = \Lambda \equiv \mu_0 e^{\frac{3\pi}{2\alpha_0}}, \tag{6}$$

independently of the choice of μ_0. Quantum electrodynamics becomes strongly coupled near the scale Λ. It is convenient to express $\alpha(\mu)$ in terms of the physical $\alpha_{phys} \approx \frac{1}{137}$ which we measure. Consider



$$\propto \frac{1}{k^2}\frac{e^2(\mu)}{1 - \Pi(k^2)}. \tag{7}$$

At the one-loop level,

$$\Pi(k^2) = -\frac{e^2(\mu)}{\pi^2}\int_0^1 dx\, x(1-x)\left(\frac{1}{\epsilon} - \frac{1}{2}\log\left(\frac{D}{\tilde{\mu}^2}\right)\right) - (Z_3 - 1), \tag{8}$$

where $\tilde{\mu}^2 \equiv \frac{4\pi\mu^2}{e^\gamma}$, $D \equiv m^2 + x(1-x)k^2$ and $m^2 = m_e^2 + O(\alpha)$, where m_e is the physical electron mass. It is convenient to introduce

$$\hat{\alpha}(k) \equiv \frac{\alpha(\mu)}{1-\Pi(k^2)} = \frac{\alpha_B}{1-\Pi_B(k^2)}. \tag{9}$$

Note that this quantity is finite, although the numerator and denominator of the last term are divergent. $\hat{\alpha}(k)$ is an effective k-dependent coupling. In particular,

$$\alpha_e = \hat{\alpha}(k=0) \approx \frac{1}{137}. \tag{10}$$

Now,

$$\Pi_{\overline{MS}}(k^2) = \frac{2\alpha(m)}{\pi}\int_0^1 dx\, x(1-x)\log\frac{D}{\mu^2}, \tag{11}$$

and so

$$\Pi_{\overline{MS}}(k^2=0) = \frac{e^2(\mu)}{6\pi^2}\log\frac{m_e}{\mu} = \frac{2\alpha(\mu)}{3\pi}\log\frac{m_e}{\mu}. \tag{12}$$

We therefore have

$$\alpha_e = \frac{\alpha(\mu)}{1-\frac{2\alpha(\mu)}{3\pi}\log\frac{m_e}{\mu}}. \tag{13}$$

We can compare this to

$$\alpha(\mu') = \frac{\alpha(\mu)}{1-\frac{2\alpha(\mu)}{3\pi}\log\frac{\mu'}{\mu}}, \tag{14}$$

and we find

$$\alpha_e = \alpha(\mu = m_e). \tag{15}$$

Therefore, in the $\overline{\text{MS}}$ scheme,

$$\alpha(\mu) = \frac{\alpha_e}{1-\frac{2\alpha_e}{3\pi}\log\frac{\mu}{m_e}}, \tag{16}$$

and the Landau pole occurs at $\Lambda = m_e e^{\frac{3\pi}{2\alpha_e}} \approx m_e e^{5\times 137}$. We now consider $\hat{\alpha}(k) = \frac{\alpha(\mu_0)}{1-\Pi_{\overline{MS}}(k^2)}$:

1. For $k^2 \gg m_e^2$, $D \sim x(1-x)k^2$, and so

$$\Pi_{\overline{MS}}(k^2) = \frac{\alpha(\mu_0)}{3\pi}\log\frac{k^2}{\mu_0^2} + \ldots. \tag{17}$$

Therefore,

$$\hat{\alpha}(k) = \frac{\alpha(\mu_0)}{1-\frac{2\alpha(\mu_0)}{3\pi}\log\frac{k}{\mu_0}} + \ldots, \tag{18}$$

and so $\hat{\alpha}(k) \approx \alpha(\mu)$ for $\mu \approx k$. Note that this is scheme-independent: in any scheme for $\mu \gg m_e$, $\alpha(\mu) \approx \hat{\alpha}(\mu)$.

2. When $k^2 \ll m_e^2$, we have $\hat{\alpha}(k) \approx \alpha_e$, but $\alpha(\mu) \longrightarrow 0$ as $\frac{\mu}{m_e} \longrightarrow 0$. For $\mu < m_e$, $\alpha(\mu)$ differs qualitatively from the physical coupling. Physically, m_e becomes important, but that is not tracked by the MS or

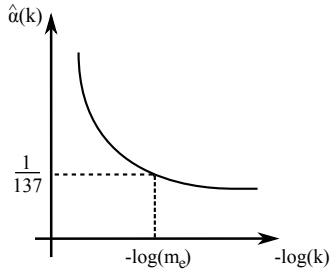


Figure 2: $\hat{\alpha}(k)$ as a function of $\log\frac{1}{k}$. At the scale of $k \sim m_e$, we have $\hat{\alpha}(m_e) \sim \frac{1}{137}$.

$\overline{\text{MS}}$ schemes: $\alpha(\mu)$ is no different for a theory with $m_e = 0$. It is more transparent to understand the behaviour of $\hat{\alpha}(k)$ using the Wilsonian approach. The coupling in the Wilsonian action, by definition, should track $\hat{\alpha}(k)$ closely.

Below the scale of m_e, the electron becomes heavy, and we can then integrate it out, leaving a pure Maxwell theory, in which the coupling constant does not run.

3. For massless quantum electrodynamics, with $m_e = 0$, $\alpha(\mu)$ is qualitatively correct for $\mu \longrightarrow 0$. We then find that $\alpha_{eff}(k) \longrightarrow 0$ as $k \longrightarrow 0$: The theory is marginally irrelevant.

5.3.4: Beta-Function of Quantum Chromodynamics

The Lagrangian of quantum chromodynamics, with an $SU(N_c)$ gauge group and N_f quarks, where $N_c = 3$ and $N_f = 6$, is given by

$$\mathscr{L} = -\frac{1}{4}\text{Tr}F_{\mu\nu}F^{\mu\nu} - i\sum_{j=1}^{N_f}\bar{\psi}_j\left(\gamma^\mu D_\mu - m_j\right)\psi_j, \tag{19}$$

with

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right],$$

$A_\mu = A_\mu^a t^a$, $F_{\mu\nu} = F_{\mu\nu}^a t^a$, where t^a are the generators of the fundamental representation of $SU(N_c)$. The covariant derivative is given by

$$D_\mu\psi_j = \partial_\mu\psi_j - igA_\mu\psi_j. \tag{20}$$

We redefine $A_\mu \longrightarrow \frac{1}{g}A_\mu$, and so the Lagrangian becomes

$$\mathscr{L} = -\frac{1}{4g^2}\text{Tr}F_{\mu\nu}F^{\mu\nu} - i\sum_{j=1}^{N_f}\bar{\psi}_j\left(\gamma^\mu D_\mu - m_j\right)\psi_j, \tag{21}$$

with

$$D_\mu\psi_j = \partial_\mu\psi_j - iA_\mu\psi_j. \tag{22}$$

We will outline the computation of the $\beta-$function for the pure gauge theory using the language of the Wilsonian approach, in the Euclidean case. Consider that the theory is provided with a cut off Λ, and bare coupling $g = g(\Lambda)$. We now write

$$A_\mu = A_\mu^{(\Lambda')} + \tilde{A}_\mu, \tag{23}$$

where $A_\mu^{(\Lambda')}$ is the part below the scale Λ', and $\tilde{A}_\mu$ is the high-energy part, above the scale Λ'. Then we have

$$S_{YM}\left[A_\mu\right] = S_{YM}\left[A_\mu^{(\Lambda')}\right] + S_1\left[A_\mu^{(\Lambda')}, \tilde{A}_\mu\right], \tag{24}$$

and

$$\begin{aligned}\int \mathfrak{D}A_\mu e^{-S[A_\mu]} &= \int \mathfrak{D}A_\mu^{(\Lambda')} e^{-S\left[A_\mu^{(\Lambda')}\right]} \int \mathfrak{D}\tilde{A}_\mu e^{-S_1\left[A_\mu^{(\Lambda')}, \tilde{A}_\mu\right]} \\ &= \int \mathfrak{D}A_\mu^{(\Lambda')} e^{-S\left[A_\mu^{(\Lambda')}\right] - \Delta S\left[A_\mu^{(\Lambda')}\right]}.\end{aligned}$$

We take the derivative expansion of ΔS,

$$\Delta S \approx c\log\frac{\Lambda}{\Lambda'}\int d^4x\, F_{\Lambda'}^2 + \ldots, \tag{25}$$

giving

$$\frac{1}{g^2(\Lambda')} = \frac{1}{g^2(\Lambda)} + c'\log\frac{\Lambda}{\Lambda'}, \tag{26}$$

where c is a pure $g-$independent number. A precise calculation gives $c = -\frac{1}{(4\pi)}\frac{22}{3}N_C$. So, for the $\beta-$function, we find

$$\beta_g = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}N_C - \frac{2}{3}N_F\right). \tag{27}$$

Considering the fermionic sector, we have

$$\begin{aligned}\int \mathfrak{D}\bar{\psi}\mathfrak{D}\psi\, e^{-\int d^dx\,\bar{\psi}(\not{D}_{\Lambda'}-m)\psi+\ldots} &\propto \det\left(\not{D}_{\Lambda'}-m\right)\\ &= e^{\log\det\left(\not{D}_{\Lambda'}-m\right)}.\end{aligned}$$

If we define $\alpha_s = \frac{g^2}{4\pi}$, we find

$$\beta_\alpha = -\frac{\alpha^2}{2\pi}\left(\frac{11}{3}N_C - \frac{2}{3}N_F\right) = -b\alpha^2, \tag{28}$$

as we found in the $g\phi^3$ theory. So, we have

$$\frac{1}{\alpha_s(\mu)} - \frac{1}{\alpha_s(\mu')} = b\log\frac{\mu}{\mu'}, \tag{29}$$

and hence

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1+\alpha_s(\mu_0)b\log\frac{\mu}{\mu_0}}. \tag{30}$$

The Landau pole occurs at

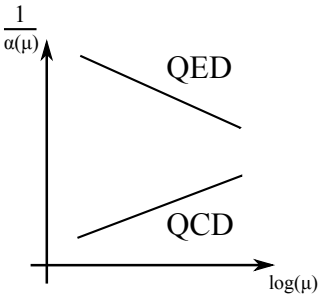


Figure 3: α^{-1} scales linearly with $\log(\mu)$ with a positive gradient in quantum chromodynamics, as with the $g\phi^3$ theory.

$$\alpha_s(\mu_0)b\log\frac{\Lambda_{QCD}}{\mu_0} = -1, \tag{31}$$

and so

$$\Lambda_{QCD} = \mu_0 e^{-\frac{1}{b\alpha_s(\mu_0)}} \approx 250\text{MeV}, \tag{32}$$

independently of our choice of μ_0. Finally, we put our coupling constant in the form

$$\alpha_s(\mu) = \frac{1}{b\log\frac{\mu}{\Lambda_{QCD}}}. \tag{33}$$

Near Λ_{QCD}, quantum chromodynamics becomes strongly coupled. The form of this coupling leads to many interesting phenomena, including confinement, and chiral symmetry breaking: $\langle\bar{U}_L U_R\rangle \neq 0$.

8.324 – Fall 2005
B. Zwiebach

NOTES ON LIE ALGEBRAS

Lie Algebra $\mathcal{G}$. A vector space $\mathcal{G}$ with a bilinear operation $[\ ,\ ]: \mathcal{G}\times\mathcal{G}\to\mathcal{G}$ such that

(i): $[x,x]=0$, for all $x\in\mathcal{G}$ (antisymmetry),

(ii) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$, for all $x,y,z\in\mathcal{G}$ (Jacobi identity).

Typically the vector space is over the real numbers (a real vector space) or over the complex numbers (a complex vector space).

A **Lie subalgebra of $\mathcal{G}$** is a vector subspace of $\mathcal{G}$ which is itself a Lie algebra under $[\ ,\ ]$.

The **generators** T_a of $\mathcal{G}$ with $a=1,2,\ldots,d$ are a set of basis vectors in $\mathcal{G}$. Here d is the dimension of $\mathcal{G}$. The Lie algebra is defined if we give the Lie brackets $[T_a,T_b]$ of all the generators. One writes

$$[T_a,T_b]=f_{ab}{}^c\,T_c\,, \tag{1}$$

where the **structure constants** $f_{ab}{}^c$ are real if the Lie algebra is a real vector space, or complex if the Lie algebra is a complex vector space.

An **ideal $\mathcal{I}$ of $\mathcal{G}$** is an invariant subalgebra of $\mathcal{G}$, namely, $[\mathcal{G},\mathcal{I}]\subset\mathcal{I}$. An ideal is **proper** if it is not equal to $\{0\}$ nor to $\mathcal{G}$ (both of which are trivial ideals of $\mathcal{G}$). The quotient space $\mathcal{G}/\mathcal{I}$ is readily checked to be a Lie algebra.

The **derived algebra** $\mathcal{G}^{(1)}$ of $\mathcal{G}$ is the set of all linear combinations of brackets of $\mathcal{G}$. We write $\mathcal{G}^{(1)}\equiv[\mathcal{G},\mathcal{G}]$. $\mathcal{G}^{(1)}$ is an ideal of $\mathcal{G}$. One defines $\mathcal{G}^{(i+1)}=[\mathcal{G}^{(i)},\mathcal{G}^{(i)}]$ for $i\geq 1$. In the **derived series** of ideals $\mathcal{G}^{(1)},\mathcal{G}^{(2)},\ldots$ each term is an ideal of $\mathcal{G}$ (this is proven using induction and the Jacobi identity). $\mathcal{G}$ is **solvable** if its derived series ends up with $\{0\}$.

If $\mathcal{I}$ and $\mathcal{J}$ are ideals of $\mathcal{G}$ then $\mathcal{I}+\mathcal{J}$ is also an ideal of $\mathcal{G}$. If, in addition both $\mathcal{I}$ and $\mathcal{J}$ are solvable, then $\mathcal{I}+\mathcal{J}$ is also solvable (show by induction that $(\mathcal{I}+\mathcal{J})^{(n)}\subset\mathcal{I}^{(n)}+\mathcal{J}$. Since $\mathcal{I}$ is solvable, at some stage the derived series of $(\mathcal{I}+\mathcal{J})$ goes into the derived series of $\mathcal{J}$, which also terminates).

The **radical** $\mathcal{G}_r$ of a Lie algebra $\mathcal{G}$ is the maximal solvable ideal of $\mathcal{G}$, *i.e.* one enclosed in no larger solvable ideal. It follows from the above additivity property that $\mathcal{G}_r$ is unique.

The **center** $\mathcal{Z}$ of $\mathcal{G}$ is the set of all elements of $\mathcal{G}$ that have zero bracket with all of $\mathcal{G}$. The center of $\mathcal{G}$ is clearly an ideal of $\mathcal{G}$.

An **abelian** Lie algebra $\mathcal{G}$ is a Lie algebra whose derived algebra $\mathcal{G}^{\{1\}} \equiv [\mathcal{G}, \mathcal{G}]$ vanishes (the Lie bracket of any two elements of $\mathcal{G}$ is always zero). For arbitrary $\mathcal{G}$, the quotient $\mathcal{G}/\mathcal{G}^{\{1\}}$ is an abelian Lie algebra. There is a unique one-dimensional Lie algebra, the abelian algebra u_1 with a single generator T and bracket $[T, T] = 0$. Any d dimensional abelian Lie algebra is (isomorphic to) the d-fold direct sum of one-dimensional Lie algebras.

The **direct sum** $\mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \cdots$ of Lie algebras with brackets $[\,,\,]_1$, $[\,,\,]_2$ $\cdots$ is the sum of the vector spaces with a bracket $[\,,\,]$ defined as: (i) $[\,x\,,\,y\,] = [\,x\,,\,y\,]_i$ when $x, y \in \mathcal{G}_i$, and (ii) $[\mathcal{G}_i, \mathcal{G}_j] = 0$, for $i \neq j$. Each summand is an ideal of the direct sum.

A Lie algebra $\mathcal{G}$ is **simple** if it has no proper ideals and is not abelian. Note the following obvious consequences:

(i) the derived algebra of $\mathcal{G}$ (an ideal) must equal $\mathcal{G}$; $\mathcal{G}$ is not solvable,

(ii) being not solvable and having no proper ideals, its radical $\mathcal{G}_r$ vanishes,

(iii) the center of $\mathcal{G}$ must vanish,

(iv) $\mathcal{G}$ cannot be broken into two sets of commuting generators.

A Lie algebra $\mathcal{G}$ is **semisimple** if its radical $\mathcal{G}_r$ vanishes. Simple Lie algebras are semisimple. It can be shown that semisimple algebras are **direct sums** of simple Lie algebras.

A **reductive** Lie algebra is the direct sum of an abelian algebra and a semisimple algebra, with both nonvanishing. This is the case of interest for non-abelian gauge theory. In these algebras the radical equals the center (the abelian algebra).

Comments. The general Lie algebra $\mathcal{G}$ is either solvable or not solvable. The solvable algebras are not easy to classify. If the algebra $\mathcal{G}$ is not solvable then either the radical vanishes, in which case the algebra is semisimple, or the radical does not vanish, in which case the quotient $(\mathcal{G}/\mathcal{G}_r)$ is semisimple (it can be shown that it has zero radical). Any Lie algebra $\mathcal{G}$ has a Levi decomposition $\mathcal{G} = P \oplus_\sigma \Lambda$, as the **semidirect sum** of a solvable algebra P and a semisimple algebra Λ. In the semidirect sum the bracket of elements within summands are the brackets of the respective algebras, and the bracket of mixed elements are defined using a representation σ of Λ on the vector space of P.

Example: The Poincare algebra $\mathcal{P}$. It has two familir subalgebras spanning together $\mathcal{P}$ as a vector space; the Lorentz algebra Λ (semisimple) and the translation algebra P (abelian). The Poincare algebra is not solvable since Λ is not, is not semisimple since its radical equals the non-vanishing algebra P, and is not reductive since it is cannot be written as a direct sum of a semisimple and an abelian part. One has $\mathcal{P} = P \oplus_\sigma \Lambda$.

THE CLASSICAL LIE ALGEBRAS

Let V be a vector space over a field F ($\mathbb{R}$ or $\mathbb{C}$). The **general linear** algebra $gl(V)$ is the algebra of endomorphisms (linear transformations, not necessarily invertible) of V. As a vector space over F, $\mathrm{End}(V)$ has dimension $(\dim(V))^2$. The Lie bracket is just the commutator of the linear transformations. More concretely, when V is a vector space of dimension n over F, one can think of $gl(V)$ as the algebra $gl(n,F)$ of $n\times n$ matrices with entries in F. The classical algebras fall into four families $\mathbf{A}_\ell, \mathbf{B}_\ell, \mathbf{C}_\ell$ and $\mathbf{D}_\ell$. These are all subalgebras of $gl(V)$. Any subalgebra of $gl(V)$ is called a **linear** Lie algebra.

$\mathbf{A}_\ell$: Let $\dim V = \ell+1$, the algebra is called $sl(\ell+1,F)$ (for special linear) and is that of endomorphisms of zero trace (a basis independent restriction). Its dimension is $(\ell+1)^2-1$.

To describe the next three families of algebras we will use bilinear forms $f(v,w)$ on an n-dimensional vector space V:

$$f(v,w) = v^t s\, w\ , \tag{2}$$

where, for a chosen basis, s is a fixed invertible $n\times n$ matrix and $v,w\in V$. We then consider V endomorphisms x such that for all v and w

$$f(x(v),w) = -f(v,x(w)) \quad \rightarrow \quad sx = -x^t s\,. \tag{3}$$

We claim that the set of endomorphisms that satisfy (3) form a Lie algebra under commutation. Indeed, if x_1 and x_2 satisfy $f(x_i(v),w) = -f(v,x_i(w))$ so does $[x_1,x_2]$. Additionally, tracing the relation $sxs^{-1} = -x^t$ we deduce that x has zero trace. The Lie algebra in question is thus a subalgebra of $sl(n,F)$. The bilinear form $f(v,w)$ is symmetric (antisymmetric) under the exchange of v and w if s is a symmetric (antisymmetric) matrix.

$\mathbf{C}_\ell$: Let $\dim V = 2\ell$ and choose some specific basis in which

$$s = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}\ . \tag{4}$$

The Lie algebra of endomorphisms that satisfy (3) is called $sp(2\ell,F)$ (for symplectic). Also, $sp(2\ell,F)\subset sl(2\ell,F)$. For a general matrix $x\in sp(2\ell,F)$ we find that the $\ell\times\ell$ blocks take the form

$$x = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \quad \rightarrow \quad n=n^t,\ p=p^t,\ m^t=-q\,. \tag{5}$$

The dimension is readily found; we need two symmetric $\ell\times\ell$ matrices p and n giving $\ell(\ell+1)$, and one arbitrary matrix m (that determines q) giving ℓ^2. Thus $\dim sp(2\ell,F) = \ell(2\ell+1)$.

$\mathbf{B}_\ell$: Let dim $V = 2\ell + 1$. This time

$$s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{pmatrix} . \tag{6}$$

The Lie algebra of V endomorphisms x that satisfy (3) is called $o(2\ell+1, F)$ (for orthogonal). Note that $o(2\ell + 1, F) \subset sl(2\ell + 1, F)$. Moreover, dim $o(2\ell + 1, F) = \ell(2\ell + 1)$ (the same dimension as that of $sp(2\ell, F)$).

$\mathbf{D}_\ell(\ell \geq 2)$: Let dim $V = 2\ell$. This time

$$s = \begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix} . \tag{7}$$

The Lie algebra of V endomorphisms x that satisfy (3) is called $o(2\ell, F)$. Note that $o(2\ell, F) \subset sl(2\ell, F)$ and that dim $o(2\ell, F) = \ell(2\ell - 1)$.

The above description given for the orthogonal algebras actually correspond to the maximal noncompact forms. The real orthogonal algebra $o(\ell)$ is obtained with a real vector space V with dim$V = \ell$ and $s = I_\ell$. This is the algebra of real antisymmetric matrices $x = -x^t$.

COMPLEX AND REAL LIE ALGEBRAS

The relevant issues are clarified with an example involving Lie algebras with three generators. Consider the Lie algebra $sl(2, C)$ of complex traceless 2×2 matrices. The Lie bracket is commutator (preserves tracelessness). This is naturally a vector space over the complex numbers as traceless matrices remain traceless by multiplication by a complex number. The algebra is therefore a complex Lie algebra. We can choose a basis of generators

$$J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \quad J_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{8}$$

The brackets are given

$$[\, J_+ \,, J_- \,] = J_3, \quad [\, J_3 \,,\, J_\pm \,] = \pm 2 J_\pm \,. \tag{9}$$

It is possible to show this complex Lie algebra is the unique simple complex Lie algebra with three generators. Even though the algebra is complex we can easily get a real algebra since the brackets in (9) have only real numbers. We can declare the vector space to be real and say that the abstract basis vectors (J_+, J_-, J_3) have the brackets in (9). This is now a

real Lie algebra. More concretely we can define the real algebra as the algebra $sl(2,R)$ of traceless 2×2 real matrices (naturally a real vector space). The generators in (8) are good ones for this algebra.

There is a way to construct another real algebra from $sl(2,C)$, this time take

$$2\,S_1 = -i(J_+ + J_-) = -\,i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad 2\,S_2 = J_- - J_+ = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \\ 2\,S_3 = -iJ_3 = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{10}$$

We recognize $S_k = -i\sigma_k/2$, with σ_k the (hermitian and traceless) Pauli matrices. The brackets have real structure constants

$$[\,S_i\,,\,S_j\,] = \epsilon_{ijk}\,S_k\,, \tag{11}$$

Declaring the S_k to be basis vectors of a real vector space we get the familiar simple real Lie algebra $su(2)$, described as the algebra of 2×2 traceless *antihermitian* matrices. Although the matrices have complex entries, the vector space is naturally real – complex multiplication ruins antihermiticity. The real algebras $sl(2,R)$ and $su(2)$ are *not* isomorphic over the reals, they are the two real forms associated with $sl(2,C)$.

The real forms of $sl(\ell+1,C)$ are $sl(\ell+1,R)$ and $su(\ell+1)$, defined as the algebra of traceless antihermitian $(\ell+1)\times(\ell+1)$ matrices (a compact subalgebra of $sl(\ell+1,C)$).

The real Lie algebra $u(\ell)$ is defined as the algebra of $\ell\times\ell$ antihermitian (complex) matrices. It has real dimension ℓ^2 (the associated Lie group is the unitary matrix group $U(\ell)$).

The real Lie algebra $usp(2\ell)$ is the algebra of antihermitian matrices in $sp(2\ell,C)$. Using (5) the antihermiticity gives

$$\begin{pmatrix} m & n \\ -n^\dagger & -m^t \end{pmatrix} \quad \text{with} \quad m^\dagger = -m,\; n^t = n\,. \tag{12}$$

The real dimension of $usp(2\ell)$ is $\ell(2\ell+1)$. The algebras $usp(2\ell)$ and $sp(2\ell,R)$ are two real forms of $sp(2\ell,C)$. Actually $usp(2\ell)$ is a compact real form while $sp(2\ell,R)$ is a non-compact real form.

REPRESENTATIONS

A **representation** of a Lie algebra $\mathcal{G}$ on a vector space V is a linear map $\psi : \mathcal{G} \to gl(V)$ which is a homomorphism of Lie-algebras ($\psi([x_1, x_2]) = [\psi(x_1), \psi(x_2)]$, for all $x_1, x_2 \in \mathcal{G}$). It is an **irreducible** representation if the image of $\mathcal{G}$ in $gl(V)$ acts on V without a proper invariant subspace.

Schur's lemma. Let $\psi : \mathcal{G} \to gl(V)$ be an irreducible representation on a finite dimensional complex vector space V. Let $\alpha : V \to V$ be a linear mapping commuting with $\psi(x)$ for all $x \in \mathcal{G}$. Then α is proportional to the identity map: $\alpha = aI$ for some number a. *Proof*: α must have one eigenvector in V with some eigenvalue a since $\det(\alpha - \lambda I) = 0$ must have at least one (complex) root $\lambda = a$). Then show that the set of vectors in V with α eigenvalue a form a nonvanishing invariant subspace. Conclude that the invariant subspace must be the whole V, so $\alpha(v) = av$ for any vector in V, which means α is a multiple of the identity.).

The **adjoint** representation of the Lie algebra is a representation where the vector space in question is precisely the Lie algebra: $V = \mathcal{G}$. We write $\mathrm{ad} : \mathcal{G} \to gl(\mathcal{G})$ and define $\mathrm{ad}\ x$ as the linear map

$$\mathrm{ad}\ x : y \to [x, y\,] \quad \text{or} \quad \mathrm{ad}\ x(y) = [x, y\,]\,. \tag{13}$$

To verify it is a representation we must check that the linear maps satisfy

$$[\,\mathrm{ad}\ x\,, \mathrm{ad}\ y\,] = \mathrm{ad}\ [x, y\,]\ . \tag{14}$$

This is verified acting on an element of the algebra and using the Jacobi identity.

The adjoint representation of a **simple** Lie algebra is irreducible for otherwise, by (13), the invariant subspace would be an ideal. For semisimple algebras the adjoint representation is reducible.

The **kernel** of ad is formed by the elements x of the algebra that generate the zero map, and therefore must have zero brackets with everything. So the kernel of ad coincides with the center $\mathcal{Z}$ of the Lie algebra. If $\mathcal{G}$ is simple its center vanishes and the map $\mathrm{ad} : \mathcal{G} \to gl(\mathcal{G})$ is one to one; any simple Lie algebra $\mathcal{G}$ is isomorphic to a subalgebra of $gl(\mathcal{G})$.

Looking at the generators we have

$$\mathrm{ad}\ T_a\,(T_b) = [T_a, T_b] = \ T_c\, f_{ab}{}^c\,. \tag{15}$$

To find the explicit matrix form of the adjoint action we view T_b as the column vector with all entries equal to zero except for a 1 in the b-th entry. Then $\mathrm{ad}\, T_a\,(T_b) = T_c(\mathrm{ad}\, T_a)_{cb}$ and we thus conclude that

$$(\mathrm{ad}\ T_a)_{cb} = f_{ab}{}^{c}\,. \tag{16}$$

An arbitrary representation r of $\mathcal{G}$ of dimension d_r is defined by $d_r \times d_r$ matrices t_a^r that represent the generators:

$$[\, t_a^r\,, t_b^r\,] = f_{ab}{}^{c}\, t_c^r\,. \tag{17}$$

On the Lie algebra there is a well defined symmetric bilinear form $\kappa\,(\cdot\,,\,\cdot)$ called the **Killing metric** and defined as (with $\circ$ denoting composition of linear maps)

$$\kappa\,(x,y) = -\mathrm{Tr}\,\Big(\mathrm{ad}\, x \circ \mathrm{ad}\, y\Big)\,. \tag{18}$$

$\kappa(x,y)$ is simply minus the trace of the operator $[\,x,[\,y,\cdot\,]\,]$. This metric is uniquely defined, up to a multiplicative constant, by two properties:
(i) invariance under automorphisms σ of the Lie algebra ($\sigma : \mathcal{G} \to \mathcal{G}$ is an automorphism if $[x,y] = z$ implies that $[\sigma(x),\sigma(y)] = \sigma(z)$. σ defines an element in $gl(\mathcal{G})$.)

$$\kappa(\sigma(x),\sigma(y)) = \kappa(x,y)\ , \tag{19}$$

(ii) associativity, in the sense that

$$\kappa(\,[x,y]\,,z\,) = \kappa\,(\,x,[y,z]\,)\,. \tag{20}$$

One readily verifies that (18) satisfies (19) (note that $\mathrm{ad}\,\sigma(x) = \sigma \circ \mathrm{ad}\, x \circ \sigma^{-1}$) and (20).

A fundamental result of Killing and Cartan is that a Lie algebra is **semisimple** if and only if its Killing form is **nondegenerate**. A bilinear form $\kappa\,(\cdot\,,\,\cdot)$ is degenerate if there is a non-zero vector x such that $\kappa\,(\,x\,,\cdot\,) = 0$. A particular case is readily verified; algebras with *abelian* ideals (thus not semisimple) have degenerate Killing forms. To see this split the algebra generators into two groups : those in the abelian ideal I and those outside I. With $x \in I$, $\kappa(x,y) = 0$ for all y or, equivalently, the trace of the operator $[x,[y,\cdot\,]\,]$ vanishes. This is clear because acting on elements of I this operator gives zero, and acting on elements outside I it gives elements inside I. Thus κ is degenerate.

A Lie algebra is said to be **compact semisimple** if the Killing form is **positive definite**, that is $\kappa(x,x) > 0$ for $x \neq 0$. The corresponding Lie group is a compact manifold.

While the product of elements $x, y \in \mathcal{G}$ is not defined, having representations $\psi(x), \psi(y)$ we can multiply them together and the Lie bracket becomes a commutator. This suggests the definition of the **universal enveloping algebra** $U(\mathcal{G})$ associated with the Lie algebra $\mathcal{G}$. It is the associative algebra spanned by monomials in the generators of $\mathcal{G}$, (say, $x, y, x^2, xy, yx, \cdots$) where monomials are identified upon use of the brackets as commutators (if $[x, y] = z$, we can say that in the enveloping algebra $xy = yx + z$).

A representation of a Lie **group** G on a vector space V is a map $G \to GL(V)$ which is a homomorphism of Lie groups. Here the **general linear group** $GL(V)$ is the group of *invertible* linear maps $V \to V$. Concretely, a representation r of dimension d_r a Lie group G is defined by an invertible $d_r \times d_r$ matrix $D^r(g)$ for each $g \in G$ such that

$$D^r(g_1)D^r(g_2) = D^r(g_1 g_2), \qquad \text{for all} \quad g_1, g_2 \in \mathcal{G}\,. \tag{21}$$

Given a Lie group G with Lie algebra $\mathcal{G}$ the **adjoint representation of the group** is a map Ad: $G \to \mathrm{Aut}(\mathcal{G})$ that associates to each element $g \in G$ an invertible linear transformation $\mathrm{Ad}\ g$ which is an automorphism of the Lie algebra $\mathcal{G}$:

$$\mathrm{Ad}\, g\,[\,x\,,\,y\,] = [\,\mathrm{Ad}\, g(x), \mathrm{Ad}\, g\,(y)\,]\,. \tag{22}$$

It follows immediately from this and (19) that the Killing form is Ad-invariant:

$$\kappa\,(\,\mathrm{Ad}\, g(x), \mathrm{Ad}\, g(y)\,) = \kappa\,(x, y\,)\,. \tag{23}$$

As befits a representation, one must have

$$\mathrm{Ad}\,(g_1 g_2) = \mathrm{Ad}\,(g_1)\mathrm{Ad}\,(g_2)\,. \tag{24}$$

The representation can be described concretely as (with repeated indices summed)

$$\mathrm{Ad}\, g : T_a \;\to\; \mathrm{Ad}\, g\,(T_a) = \; T_b\, \bar{D}_{ba}(g)\,, \tag{25}$$

where $\bar{D}_{ab}(g)$ denotes $\dim(\mathcal{G})\times\dim(\mathcal{G})$ matrix representation of $\mathrm{Ad}\, g$. One readily checks that the above two equations imply that, as expected, $\bar{D}_{ab}(g_1 g_2) = \bar{D}_{ac}(g_1)\bar{D}_{cb}(g_2)$.

Geometrically one understands the adjoint group action on the algebra as follows. Recall that the Lie algebra $\mathcal{G}$ associated with a Lie group G can be identified with the tangent space of G at the identity. For any fixed $g \in G$ the transformation $h \to ghg^{-1}$ (conjugation by g)

is a map of G to itself leaving the identity element fixed and inducing a linear transformation from the tangent space at h to the tangent space at ghg^{-1}. Therefore, this map induces a linear transformation on the tangent space at the identity, a linear transformation on the Lie algebra[1].

The adjoint action can be described using any $\mathcal{G}$ representation and a G representation of the same dimensionality. Let t_a^r be matrices representing the generators T_a in a representation r of $\mathcal{G}$ and let $D^r(g)$ be matrices of the same dimension in the representation r of G. The action of Ad g is via conjugation with $D^r(g)$, and (25) gives

$$D^r(g)\, t_a^r\, D^r(g)^{-1} = t_b^r\, \bar{D}_{ba}(g)\ . \tag{26}$$

If the representation r of $\mathcal{G}$ is the adjoint, $D^r(g) = \bar{D}(g)$. Note that the $\bar{D}_{ba}(g)$ on the above right-hand side are numbers, not matrices.

For a matrix group G whose elements $V \in G$ are $k \times k$ matrices, the generators T_a of the associated Lie algebra $\mathcal{G}$ are themselves $k \times k$ matrices (this is called the fundamental representation) and (25) and (26) read

$$\mathrm{Ad}\, V(T_a) = V\, T_a V^{-1} = T_b\, \bar{D}_{ba}(V)\,. \tag{27}$$

COMPACT SEMISIMPLE LIE ALGEBRAS

Consider again the Killing metric

$$\kappa_{ab} \equiv \kappa\,(T_a, T_b) = -\mathrm{Tr}\,(\,\mathrm{ad}\, T_a \circ \mathrm{ad}\, T_b\,) = -\mathrm{Tr}\,(\,\bar{t}_a\, \bar{t}_b\,)\,. \tag{28}$$

where $\bar{t}_a$ denotes the matrix representation of the adjoint action (ad T_a). By (16) we have $(\bar{t}_a)_{cb} = f_{ab}{}^c$. The metric is real since the structure constants are real. We note that

$$-\mathrm{Tr}\,(\,[\,\bar{t}_a\,,\,\bar{t}_b\,]\,\bar{t}_c\,) = f_{ab}{}^e\,\kappa_{ec}\,. \tag{29}$$

Cyclicity of the trace implies that the $f_{ab}{}^e\,\kappa_{ec}$ is totally antisymmetric in a, b and c.

[1]More precisely the Lie Algebra $\mathcal{G}$ associated with a group G is defined as the set of all left-invariant vector fields on the group manifold. The bracket is the Lie bracket of vector fields, viewed as differential operators. As a vector space, the set of left-invariant vector fields is isomorphic to the tangent space to the group at the identity (each tangent vector at the identity can be extended to a left-invariant vector field). By left-invariant one means invariant under diffeomorphisms of the group induced by left multiplication: $g \to ag$, $\forall g \in G$.

Since the matrix κ_{ab} is symmetric, real, and positive definite ($\mathcal{G}$ is compact semisimple)[2] there is an real orthogonal $\mathcal{O}$ matrix that diagonalizes κ, namely $\mathcal{O}\kappa\,\mathcal{O}^T = \mathcal{D}$, with $\mathcal{D}$ a diagonal matrix with positive entries. We can define new generators $T'_a = \mathcal{O}_{ac}T_c$, and then we verify that

$$-\mathrm{Tr}\,(\ \bar{t}'_a\,\bar{t}'_b\ ) = \mathcal{D}_{ab}\,.$$

By a further (real) scaling all diagonal elements can be made equal to a constant denoted as $C(\mathcal{G})$. In the resulting basis, which we call again T_a, we have

$$-\mathrm{Tr}\,(\,\bar{t}_a\,\bar{t}_b\,) = C(\mathcal{G})\,\delta_{ab}\ = \kappa_{ab}\,. \tag{30}$$

In this basis we write $[T_a, T_b] = f_{ab}{}^c\,T_c$. Since $f_{ab}{}^e\kappa_{ec}$ is totally antisymmetric, (30) implies that $f_{ab}{}^e\delta_{ec} = f_{ab}{}^c$ is totally antisymmetric. We thus define a totally antisymmetric symbol

$$f_{abc} \equiv f_{ab}{}^c\,, \tag{31}$$

and write

$$[T_a, T_b] = f_{abc}\,T_c\,, \qquad (\mathrm{ad}\ T_a)_{bc} = -f_{abc}\,. \tag{32}$$

We have shown that for compact semisimple real Lie algebras there is a basis in which equations (30) and (32) hold.

We now verify that the adjoint representation of G acts on $\mathcal{G}$ via real orthogonal matrices. For this we begin with (30), insert $\bar{D}$ matrices, and use (26) applied to the adjoint:

$$\begin{aligned} -C(\mathcal{G})\delta_{ab} &= \mathrm{Tr}(\bar{t}_a\,\bar{t}_b) \\ &= \mathrm{Tr}\left(\bar{D}(g)\,\bar{t}_a\,\bar{D}^{-1}(g)\bar{D}(g)\,\bar{t}_b\bar{D}^{-1}(g)\right) \\ &= \mathrm{Tr}\left(\bar{t}_e\,\bar{t}_f\right)\bar{D}_{ea}(g)\bar{D}_{fb}(g) \\ &= -C(\mathcal{G})\bar{D}_{ea}(g)\bar{D}_{eb}(g)\,, \end{aligned} \tag{33}$$

showing that the matrix $\bar{D}(g)$ is indeed orthogonal ($\bar{D}^T\bar{D} = 1$).

For arbitrary representations of the Lie algebra we define the matrix $\kappa\,(r)$

$$\kappa_{ab}(r) \equiv\ -\mathrm{Tr}\,(\,t^r_a\,t^r_b\,)\,. \tag{34}$$

[2]Positivity also follows if the matrices in the adjoint representation of $\mathcal{G}$ are antihermitian: any $x \in \mathcal{G}$ is represented in the adjoint by an $X(= -X^\dagger)$ and $\kappa(x,x) = -\mathrm{Tr}(XX) = \mathrm{Tr}(X^\dagger X) \geq 0$.

Inserting group matrices $D^r(g)$ as in (33) and using (26) we see that

$$\kappa_{ab}(r) = \kappa_{cd}(r)\, \bar{D}_{ca}(g) \bar{D}_{db}(g) \quad \rightarrow \quad \kappa(r) = \bar{D}^T(g)\, \kappa(r) \bar{D}(g)\,.$$

Since $\bar{D}(g)$ is orthogonal it follows that $\kappa(r)$ commutes with all matrices $\bar{D}(g)$. For simple algebras the adjoint representation of the corresponding group acts irreducibly. By Schur's lemma this means that $\kappa(r)$ must be proportional to the identity. We thus define

$$-\mathrm{Tr}\,(\, t^r_a\, t^r_b\,) = C(r)\, \delta_{ab}\,. \tag{35}$$

In the chosen basis, the element T_aT_a (a summed!) of the enveloping algebra is called a **Casimir operator**. It commutes with all T_a's and therefore with all elements of the enveloping algebra. Indeed, with repeated indices summed,

$$[\, T_aT_a\,,\, T_c] = T_a\, (f_{acb} T_b) + \, (f_{acb}\, T_b) T_a = (f_{acb} - f_{bca}) T_a T_b = 0\,. \tag{36}$$

More geometrically, for any semisimple algebra $\kappa^{ab} T_a T_b$ is a Casimir, where the inverse κ^{ab} of the Killing metric κ_{ab} exists because of semisimplicity.

Since T_aT_a is a Casimir, in any irreducible representation r the Casimir matrix $t^r_a\, t^r_a$ commutes with all the matrices t^r_a representing generators. The Casimir matrix, by Schur's lemma, must be a multiple of the identity in *any* irreducible representation:

$$-t^r_a\, t^r_a = C_2(r)\, I_{d(r)}\,, \tag{37}$$

where $d(r)$ denotes the dimension of the representation r. The constants $C(r)$ and $C_2(r)$ are simply related; taking the trace of (37) and the contraction of (30) one finds

$$C_2(r)\, \dim(r) = C(r)\, \dim(\mathcal{G})\,. \tag{38}$$

For the adjoint representation (37) gives

$$f_{acd}\, f_{bcd} = C_2(\mathcal{G}) \delta_{ab} \;\rightarrow\; C_2(\mathcal{G}) = \frac{1}{\dim\,(\mathcal{G})}\; f_{abc}\, f_{abc}\,, \tag{39}$$

and (38) implies that $C_2(\mathcal{G}) = C(\mathcal{G})$. As defined, the constants C and C_2 are basis dependent.